

Marshall-Olkin Generalized Distributions and Their Applications

MAJOR RESEARCH PROJECT REPORT

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This project work has helped in generalizing the Marshall-Olkin distribu-

tions and developing new autoregressive time series models. We have applied the results to data sets from different contexts like industry, hydrology, biostatistics etc. The models have been applied in reliability and information theory, stress-strength analysis, acceptance sampling, entropy, record values, order statistics etc. Various autoregressive minification processes have also been developed. Maximum likelihood estimates of the parameters have been obtained and necessary R programs were developed. Thus the study has contributed to the advancement of new knowledge with applications in a wide range of contexts. The study has resulted in the publication of more than 5 research papers and a Ph.D.Thesis. A few more research papers have been communicated for publication in reputed international journals.

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Abstract and Keywords

The main objectives of this project work are concerned with the study on some generalizations of Marshall-Olkin family of distributions like Exponential, Weibull, Fréchet, Lindley, Pareto, Rayleigh etc and their applications in various areas such as time series modelling, autoregressive minification processes, reliability analysis, record value theory, acceptance sampling, etc. We consider two different generalizations of the Marshall-Olkin family namely the Exponentiated Marshall-Olkin family, Negative Binomial Extreme Stable family.

Chapter 1 is an introductory one with a review of literature and a brief summary of the work. In chapter 2 we introduce Marshall-Olkin extended exponential distribution. We consider hazard rate function, record values etc. We estimate the parameters and describe various applications of the new distribution. We derive the BLUE's of location and scale parameters of the model using upper records. Using BLUE the 95% confidence interval for the parameters are also obtained. The results are verified and the future records are predicted from simulated data sets. Stress-strength analysis is carried out and the results are verified by simulation. A minification process is developed and sample path properties are explored. Parameters of the process are estimated and the results are verified by simulation studies.

Chapter 3 is devoted to introduction of Marshall-Olkin extended Fréchet distribution and derivation of its properties. The distribution of sample extremes, Renyi entropy, order statistics, etc are obtained. The new model is then compared with the Fréchet, the exponentiated Fréchet and beta Fréchet distributions using a real data set on the survival times of injected guinea pigs and proved to be a better fit.

Chapter 4 is dedicated to various applications of Marshall-Olkin extended Fréchet distribution specifically in estimation of reliability under stress-strength model and establishing the validity of estimate through simulation, developing a suitable sampling plan for a product with life time following the new model and and four different minification processes and its properties.

In chapter 5, we concentrate on the Marshall-Olkin Exponentiated Generalized Exponential distribution and its applications. Various properties are explored. The maximum likelihood estimates are obtained and the models are applied to a real data set on carbon fibers. Stress-strength analysis with respect to the model is also carried out. R programs necessary for computation are also developed.

Chapter 6 deals with a new distribution namely, Marshall-Olkin Exponentiated Generalized Fréchet distribution and its applications. Quantiles and order statistics of the distributions are obtained. Estimates of the parameters of the distribution are obtained and applied to a real data set. Reliability of a system following the new distribution under stress-strength model is estimated and simulation studies are carried out for establishing the validity of the estimates. The R program developed is also given.

In Chapter 7, we introduce the Exponentiated Marshall-Olkin Exponential and Weibull distribution. Various properties are studied including quantiles, order statistics, record values and Renyi entropy. Estimation of parameters is also considered. A real data set is analyzed as an application. Chapter 8 deals with the Negative binomial extreme stable Marshall-Olkin extended Lindley distribution and its properties. We consider the properties of Extended Lindley distribution. The expression for quantiles and the distribution of order statistics are derived. Distribution of the record values is obtained. The maximum likelihood estimates are obtained and applied to a real data set on failure times of air conditioning system of an air plane.

Chapter 9 concentrates on Negative binomial extreme stable Marshall-Olkin Pareto distribution. We develop reliability test plans for acceptance or rejection of a lot of products submitted for inspections with lifetime following the new distribution. The results are illustrated by numerical examples. In chapter 10, we introduce a new distribution namely, the Negative binomial Marshall-Olkin Rayleigh distribution. Various properties are discussed. Maximum likelihood estimates are obtained and applied to a real data set on remission times of bladder cancer patients. The results are useful in constructing a suitable sampling plan for a product with the new distribution as lifetime.

Keywords: Acceptance sampling plan, Entropy, Exponentiated Exponential distribution, Exponentiated Marshall-Olkin Exponential distribution, Exponentiated Marshall-Olkin Weibull distribution, Extended Lindley distribution, Hazard function, Marshall-Olkin Exponentiated Generalized Exponential distribution, Marshall-Olkin Exponentiated Generalized Fréchet distribution, Negative binomial extreme stable Marshall-Olkin extended Lindley distribution, Negative binomial extreme stable Marshall-Olkin Pareto distribution, Negative binomial Marshall-Olkin Rayleigh distribution, Order statistics, Probability density function, Quantiles, Record Values, Stress-Strength analysis, Survival function.

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CHAPTER 1

Introduction and Summary

1.1 Introduction

The main objective of this research work is to study some Generalizations of Marshall-Olkin family of distributions and their applications. The study is mainly concentrated on some generalized Marshall-Olkin distributions like Exponential, Weibull, Fréchet, Lindley, Pareto, Rayleigh etc and their applications in various areas of statistical theory such as reliability analysis, record value theory, acceptance sampling, etc. Statistical modeling, simulation studies and inference are the principal areas of exploratory data analysis. Different generalizations of Marshall-Olkin family of distributions have been discussed by many authors. Various lifetime models are used for the probabilistic analysis of the lifelength of a system or a device. Such distributions are most frequently used in the fields like medicine, engineering etc. Many parametric models such as exponential, gamma, Weibull are commonly used in statistical literature to analyze lifetime data.

Exponential and Weibull distribution play central role in reliability theory and survival

analysis. Pareto distribution and Burr distribution are used for modeling and predicting a wide variety of socioeconomic variables such as size of income, wealth etc. The Rayleigh distribution is used to model wave heights in oceanography, and electromagnetic signals in communication theory. Lindley model is useful to analyze lifetime data, especially in modeling stress-strength reliability. Fréchet distribution is useful for modeling and analysis of several extreme events ranging from accelerated life testing to earthquakes, floods, rainfall, sea currents and wind speeds.

1.2 Review of Literature

Adding parameters to a well-established distribution is a time honored device for obtaining more flexible new families of distributions. Marshall and Olkin (2007) discussed an interesting method of adding a new parameter to an existing distribution. It includes the baseline distribution as a special case and gives more flexibility to model various types of data. Introduction of a scale parameter leads to the accelerated life model, and taking powers of the survival function introduces a parameter that leads to the proportional hazards model. For instance, the family of Weibull distributions contains the exponential distributions and is constructed by taking powers of exponentially distributed random variables. The family of gamma distributions also contains the exponential distributions, and is constructed by taking sum of independent and identically distributed(i.i.d) exponential random variables. The MO family of distributions arises in different contexts and is known under various names such as the proportional odds family or proportional odds model or family with tilt parameter or proportional hazards family or proportional reverse hazards family etc.

1.2.1 Marshall-Olkin family of Distributions

Marshall-Olkin (1997) introduced a new family of distributions by introducing a new parameter to existing family of distributions. If $\bar{F}(x)$ is a survival function, then the new family

has survival function given by

$$\bar{G}(x) = \frac{\alpha \bar{F}(x)}{1 - (1 - \alpha) \bar{F}(x)}, 0 < \alpha < \infty, -\infty < x < \infty \quad (1.2.1)$$

This family of distributions possesses very interesting properties. Let X_1, X_2, \dots be a sequence of independent and identically distributed random variables with survival function $\bar{F}(x)$ and let $U_N = \min(X_1, X_2, \dots, X_N)$ where N is a geometric random variable with $P(N = n) = \alpha(1 - \alpha)^{n-1}; 0 < \alpha < 1$, X_i 's and N are independent. Then U_N has a survival function given by (1.2.1). If $\alpha > 1$ and $V_N = \max(X_1, X_2, \dots, X_N)$ where N follows geometric distribution with $P(N = n) = \alpha^{-1}(1 - \alpha^{-1})^{n-1}$, then V_N also has the survival function given by (1.2.1). This property is usually known as geometric extreme stability. The probability function and hazard rate function corresponding to (1.2.1) can be obtained as

$$g(x, \alpha) = \frac{\alpha f(x)}{(1 - \alpha \bar{F}(x))^2} \quad (1.2.2)$$

and

$$h(x, \alpha) = \frac{r_F(x)}{(1 - \alpha \bar{F}(x))} \quad (1.2.3)$$

where $f(x)$ and $r_F(x)$ are the p.d.f. and hazard rate function corresponding to $\bar{F}(x)$. This family can be considered as exponential compounding models.

This family has connections to reliability contexts also, with respect to proportional odds models. Sankaran and Jayakumar (2008) presented the physical interpretation of Marshall-Olkin family in the context of proportional odds model. Let X be distributed with survival function $\bar{F}(x)$ and let $\underline{z} = (z_1, z_2, \dots, z_p)^T$ be a co variate vector of order p . Then the survival function of the proportional odds model is given by

$$\bar{G}(x|z) = \frac{k(z)\bar{F}(x)}{1 - (1 - k(z))\bar{F}(x)}$$

where $k(z) = \frac{\lambda_G(x)}{\lambda_F(x;z)}$, a non negative function of co-variate z , $\lambda_F(x; z)$ is the proportional odds function corresponding to $\bar{F}(x)$ and $\lambda_G(x)$ represents an arbitrary odds function with respect to $\bar{G}(x)$. When $k(z) = \alpha$, a constant, then \bar{G} reduces to the family given by (1.2.1).

Many authors have proposed various univariate distributions belonging to the Marshall-Olkin family of distributions such as Marshall-Olkin Weibull [Ghitany et al. (2005)], Marshall-Olkin Pareto [Alice and Jose (2003)], Marshall-Olkin semi-Weibull [Alice and Jose (2005)], Marshall-Olkin Lomax [Ghitany et al. (2007)], Marshall-Olkin semi Burr and Marshall-Olkin Burr [Jayakumar and Thomas (2008)] and Marshall-Olkin q-Weibull [Jose et al.(2010)]. Jose et al. (2011) consider bivariate Marshall-Olkin Weibull distributions and processes. Krishna et.al (2013a,b) introduced Marshall-Olkin Fréchet distribution and discussed a number of applications. Jose et al.(2013) discussed MarshallOlkin MorgensternWeibull distribution and its applications.

1.2.2 Exponentiated Marshall-Olkin family

Exponentiated Marshall-Olkin family was introduced by Jayakumar and Thomas (2008) as a generalization of Marshall-Olkin family of distributions and is characterized by the survival function given by

$$\bar{G}(x) = \left(\frac{\alpha\bar{F}(x)}{1 - \alpha\bar{F}(x)} \right)^\gamma; \quad \alpha > 0, \gamma > 0, x \in R \quad (1.2.4)$$

These can be regarded as Gamma compounding models. When $\gamma = 1$, $\bar{G}(x)$ becomes the Marshall-Olkin family given by (1.2.1).

The probability density function is given by

$$g(x; \alpha, \gamma) = \frac{\gamma \alpha^\gamma (\bar{F}(x))^{\gamma-1} f(x)}{[1 - \alpha \bar{F}(x)]^{\gamma+1}}; \alpha > 0, \gamma > 0$$

and hazard rate function is given by

$$h(x; \alpha, \gamma) = \frac{\gamma r_F(x)}{(1 - (1 - \alpha)\bar{F}(x))}; \alpha > 0, \gamma > 0$$

where $f(x)$ and $r_F(x)$ are the p.d.f. and hazard rate function corresponding to $\bar{F}(x)$.

1.2.3 Negative Binomial Extreme Stable family

Nadarajah et al.(2012) introduced another generalization of Marshall-Olkin family of distribution. Let X_1, X_2, \dots be a sequence of independent and identically distributed random variables with survival function $\bar{F}(x)$. Suppose N is independent of X_i 's with a truncated negative binomial distribution with probability density function

$$P(N = n) = \frac{\alpha^\gamma}{1 - \alpha^\gamma} \binom{\gamma + n - 1}{\gamma - 1} (1 - \alpha)^\gamma; \gamma > 0, 0 < \alpha < 1, n = 1, 2, \dots$$

Consider $U_N = \min(X_1, X_2, \dots, X_N)$, then

$$\begin{aligned} \bar{G}_U(x) &= \frac{\alpha^\gamma}{1 - \alpha^\gamma} \sum_{n=1}^{\infty} \binom{\gamma + n - 1}{\gamma - 1} ((1 - \alpha)\bar{F}(x))^n \\ &= \frac{\alpha^\gamma}{1 - \alpha^\gamma} \left[(F(x) + \alpha\bar{F}(x))^{-\gamma} - 1 \right] \end{aligned} \quad (1.2.5)$$

Similarly when $\alpha > 1$, consider $V_N = \max(X_1, X_2, \dots, X_N)$ where N follows truncated negative binomial random variables with parameters α^{-1} and $\gamma > 0$. Then the survival function of V_N is given by (1.2.5). This property may be referred to as Negative Binomial

Extreme Stability. The probability density function and hazard rate function corresponding to (1.2.5) is given by

$$g(x; \alpha, \gamma) = \frac{(1 - \alpha)\gamma\alpha^\gamma f(x)}{(1 - \alpha^\gamma)(F(x) + \alpha\bar{F}(x))^{\gamma+1}}; \quad x > 0, \alpha > 0, \gamma > 0$$

and

$$h(x; \alpha, \gamma) = \frac{(1 - \alpha)\gamma\bar{F}(x)r_F(x)}{(F(x) + \alpha\bar{F}(x))(1 - (F(x) + \alpha\bar{F}(x))^\gamma)}$$

where $f(x)$ and $r_F(x)$ are the p.d.f. and hazard rate function corresponding to $\bar{F}(x)$.

The new family in (1.2.5) can be interpreted as follows. Suppose the failure times of a device are observed. Every time a failure occurs the device is repaired to resume function. Suppose also that the device is deemed no longer useable when a failure occurs that exceeds a certain level of severity. Let X_1, X_2, \dots denote the failure times and let N denote the number of failures. Then U_N will represent the time to the first failure of the device and V_N will represent the lifetime of the device. Therefore, (1.2.5) could be used to model both the time to the first failure and the lifetime.

1.3 Applications

In this section we consider some applications of the new distributions introduced in the thesis. We have considered applications in the theory of record values, entropy analysis, reliability modeling, stress-strength analysis, acceptance sampling and inspection plans.

1.3.1 Record Values

Let X_1, X_2, \dots be a sequence of i.i.d random variables with common distribution function F . An observation X_j will be called a record (upper record) if it exceeds in value all preceding observations, i.e., if $X_j > X_i$, for every $i < j$. Let X_j be observed time at j . The sequence of record times $T_n; n \geq 0$ is defined as $T_0 = 1$ with probability 1 and for $n > 1, T_n =$

$\min\{j; X_j > X_{T_{n-1}}\}$. Let R_n be the record value sequence defined as $R_n = X_{T_n}; n = 0, 1, 2, \dots$ and is called n^{th} record. R_0 is the reference value or trivial record. Let $g_{R_n}(x)$ denote the probability density function of n^{th} record and is given by

$$g_{R_n}(x) = \frac{g(x)(H(x))^{n-1}}{(n-1)!}; -\infty < x < \infty \quad (1.3.1)$$

where $H(x) = -\log(1 - G(x))$.

Then the joint pdf of m^{th} and n^{th} record statistics is given by

$$g_{R_m, R_n}(x, y) = \frac{(H(x))^{m-1} (H(y) - H(x))^{n-m-1} g(x)g(y)}{(m-1)! (n-m-1)! (1-G(x))}; -\infty < x < y < \infty \quad (1.3.2)$$

where $H(x) = -\log(1 - G(x))$ and $H(y) = -\log(1 - G(y))$.

Chandler (1952) introduced the concept of record values in statistical theory. Feller (1966) gave some examples of record values with respect to gambling problems. Record values are used in industrial stress testing, meteorological analysis, sporting and athletic events, and oil and mining surveys etc. It is closely related to order statistics. The recent works of record value theory and its applications are available in Ahsanullah (1995,1997), Balakrishnan and Ahsanullah (1994), Arnold et al.(1998), Raqab (2001), Bieniek and Szy-nal (2002), Saran and Singh (2008), Jose et al.(2014) discussed applications of Marshall-Olkin family in the context of record values and reliability theory. In the thesis we consider various applications in record value theory.

1.3.2 Entropy Analysis

Entropy is a measure of uncertainty. In 1865, the German physicist Rudolf Clausius Shannon introduced the term entropy. Various entropy measures have been developed by mathematicians, engineers and Physicists to describe several phenomena, in the context of communications theory. In 1875, the Austrian physicist Ludwig Boltzmann and the Ameri-

can scientist Willard Gibbs put entropy into the probabilistic setup of statistical mechanics. In 1948, an American electrical engineer and mathematician Claude E. Shannon applied the theory of entropic functional in the theory of digital communications with the discrete form $s = -k \sum_{i=1}^n p_i \ln(p_i)$, called Shannons entropy. The concept of entropy has been successfully applied in a variety of fields including statistical mechanics, statistical information theory, stock market analysis, queuing theory, image analysis, reliability estimation, etc. (see, e.g., Kapur (1993)). Ebrahimi (2000) discussed the maximum entropy method for life time distributions. Mathai and Rathie (1975) consider various generalizations of Shannon entropy measure and describe various properties including additivity, characterization theorem etc. Mathai and Haubold (2007) introduced a new generalization of the Shannon entropy measure using a general class of distributions called pathway distributions. Jose and Shanoja (2008) showed that pathway model can be obtained by optimizing Mathai's generalized entropy with a more general setup, which is a generalization of various entropy measures due to Shannon and others. In the thesis we consider Shannon entropy, Rényi entropy etc.

1.3.3 Stress-Strength Reliability Modeling

The stress-strength reliability analysis is concerned with an assessment of reliability of a system in terms of random variables X representing stress and Y representing the strength. If the stress exceeds strength the system would fail. The term stress-strength was first introduced by Church and Harris (1970). The stress-strength reliability can be defined as $R = P(X < Y)$. Gupta et al (2010) obtained various results on the MO family in the context of reliability modeling and survival analysis. For more details see, Kotz et al.(2003), Kundu and Gupta (2005,2006), Raqab and Kundu (2005), Kundu and Raqab (2009), Bindu (2011), Krishna et al.(2013b) etc.

1.3.4 Acceptance Sampling Inspection Plans

Acceptance sampling is a statistical procedure used in quality control and it involves sampling inspection in which decisions are made to accept or reject products or services. Acceptance sampling is developed by Dodge and Roming (1959). Two types of acceptance sampling are (1) Attributes sampling, in which the presence or absence of a characteristic in the inspected item is only taken note of, and (2) Variable sampling, in which the presence or absence of a characteristic in the inspected item is measured on a predetermined scale. Kantam et al.(2001), Rosaiah et al.(2005), Rao et al.(2009), Lio et al.(2010), Krishna et al.(2013), Kantam and Sriram (2010), Jose et al.(2011) etc have discussed acceptance sampling plans for various distributions. Rao (2010) developed a group acceptance sampling plan for a truncated life test when the lifetime of an item follows Marshall-Olkin extended Lomax distribution.

1.3.5 Exponentiated Generalized Distributions

Gupta et al.(1998) introduced exponentiated exponential distribution as a generalization of standard exponential distribution. Nadarajah and Kotz (2006) proposed the exponentiated gamma, exponentiated Fréchet and exponentiated Gumbel distributions. Cordeiro et al. (2013) proposed a new class of distributions that extend the exponentiated generalized type distributions. Given a continuous cdf $G(x)$, the cdf of the Exponentiated Generalized (EG) class of distributions is of the form

$$F(x) = [1 - \{1 - G(x)\}^\alpha]^\beta \quad (1.3.3)$$

where $\alpha > 0$ and $\beta > 0$ are two additional shape parameters. The base line distribution $G(x)$ is a special case of (1.3.3) when $\alpha = \beta = 1$. When $\alpha = 1$, it reduces to the exponentiated type distributions. The probability density function of the exponentiated

generalized class of distributions is given by

$$f(x) = \alpha\beta\{1 - G(x)\}^{\alpha-1}[1 - \{1 - G(x)\}^\alpha]^{\beta-1}g(x) \quad (1.3.4)$$

The exponentiated generalized densities allow for greater flexibility of tails and can be widely applied in many areas of engineering and biology. Eugene et al.(2002) discussed the beta generalized family, which includes two extra parameters but involves the beta incomplete function. The recent work in exponentiated generalized distributions are Lemonte and Cordeiro(2011), Cordeiro et al.(2011), Sarhan et al.(2013), Elbatal and Muhammed(2014) etc.

1.3.6 Time Series Modelling

Time series modelling and analysis has become one of the most important and widely used branches of mathematical statistics. Application of this branch of statistics includes financial modelling, economic forecasting, stock market analysis, seismological studies, study of biological data, neuro physiology, astro physics and communications engineering. The experimental data that have been observed at different points in time leads to a time series. A basic assumption in any time series analysis or modeling is that some aspects of the past pattern will continue to remain in the future. Also under this setup often the time series process is assumed to be based on past values of the main variable but not on explanatory variables which may affect the variables or system. Time series models usually represents a stochastic process and can have many forms. The three important models mainly used are autoregressive models (AR), the integrated models (I), and the moving average model (MA). These three models combined to get autoregressive moving average (ARMA) and autoregressive integrated moving average (ARIMA) models. An important class of stochastic models for describing time series, which has received a great deal of attention is the so called stationary models which assume that the process remains in equilibrium about a constant mean level. More precisely, stationarity implies that the joint probability distribution of the process is invariant over time. The time series X_t is said to

be stationary if, for any $t_1, t_2, \dots, t_n \in Z$, and $n = 1, 2, \dots$,

$$F_{x_{t_1}, x_{t_2}, \dots, x_{t_n}}(x_1, x_2, \dots, x_n) = F_{x_{t_1+k}, x_{t_2+k}, \dots, x_{t_n+k}}(x_1, x_2, \dots, x_n)$$

where F denotes the distribution function of the set of random variables.

1.3.7 Autoregressive process

In statistics and signal processing an autoregressive model is a type of random process which is often used to model and predict various type of natural phenomena. The autoregressive model is one of a group of linear prediction formulae that attempt to predict an output of a system based on the previous observations. The notation $AR(p)$ indicates an autoregressive model of order p . The $AR(p)$ model is defined as

$$X_n = \sum_{i=1}^p a_i X_{n-i} + \varepsilon_n$$

where a_1, \dots, a_p are parameters of the model, $a_p \neq 0$ and $\{\varepsilon_n\}$ an innovation process of independently and identically distributed random variables to ensure that $\{X_n\}$ is a stationary Markov process with a specified marginal distribution function $F(x)$. In the literature most processes are assumed to have gaussian distribution. See (Van Trees (1968)). This is widely preferred, since most parameter estimation techniques can lead to analytically tractable solutions under this assumption. More over this assumption has been based on the central limit theorem and is valid for processes having finite variance. Therefore processes having infinite variances can not be modeled as gaussian. Such processes are called non-gaussian processes and are represented by other distributions in the literature. There is also emperical evidence that many real life time series are non-Gaussian and have structure that change over time.

Several authors introduced non Gaussian process with specified marginals. Tavares (1980) introduced the exact distribution of extremes of a non-Gaussian process, Gawer

and Lewis (1980) developed first order autoregressive gamma sequences and point processes. Brown et al (1984) described the need for constructing time series models with Weibull marginal distribution for the modeling of wind velocity data. Gibson (1986) describes the use of autoregressive processes with Laplace marginal distribution for image source modeling. Yeh et al (1988) developed Pareto process. Pillai (1991) studied semi-Pareto processes. Jayakumar and Pillai (1993) introduced autoregressive Mittag-leffler process. Anderson and Arnold (1993) described the use of Linnik marginal distribution in modeling stock price returns and other financial data. Balakrishna (1998) considered estimation for the semi-Pareto process. Nielson and Shephard (2003) discussed likelihood analysis of a first order autoregressive model with exponential innovations. Jayakumar and Thomas (2004) studied semi logistic process and Jayakumar (2006) developed an autoregressive model for Asymmetric Laplace distribution.

1.3.8 Minification Process

Minification process is another non-linear autoregressive model available in literature. Tavares (1980) introduced a minification process of the form

$$X_n = k \min(X_{n-1}, \varepsilon_n), n \geq 1$$

where $k \geq 1$ and $\{\varepsilon_n, n \geq 1\}$ is an innovation process of identically and independently distributed random variables. Lewis and McKenzie (1991) introduced first order minification process having the structure

$$\begin{aligned} X_n &= k X_{n-1} \text{ with probability } p \\ &= k \min(X_{n-1}, \varepsilon_n) \text{ with probability } 1-p \end{aligned}$$

where $\{\varepsilon_n\}$ is a sequence of identically and independently distributed random variables independent of X_n . Another form of minification process is the one having the structure

$$\begin{aligned} X_n &= k \varepsilon_n \text{ with probability } p \\ &= k \min(X_{n-1}, \varepsilon_n) \text{ with probability } 1-p \end{aligned}$$

Sim(1994) introduced the minification process with Weibull marginal distribution. Arnold and Robertson(1989)introduced a logistic minification process. Pillai et al(1995) discussed autoregressive minification process and universal geometric minima. Balakrishna and Jacob(2003) estimated the parameters of minification process. Alice and Jose (2003) introduced Marshall-Olkin Pareto process. Alice and Jose(2005) developed Marshall-Olkin semi-Weibull minification process. Some bivariate and multivariate minification processes are introduced and studied by Balakrishnan and Jayakumar(1997), Thomas and Jose (2002,2004), Ristic(2006), Thomas and Jayakumar (2008), Naik and Jose (2008), Ancy et al.(2009)

1.3.9 Autocorrelation and Auto Covariance

Auto correlation is the internal correlation of the observations in a time series, usually expressed as a function of the time lag between observations. The autocorrelation at lag k , $\gamma(k)$ is defined mathematically as

$$\gamma(k) = \frac{E(X_t - \mu)(X_{t+k} - \mu)}{E(X_t - \mu)^2}$$

where $X_t, t = 0, \pm 1, \pm 2, \dots$ represent the values of series and μ is the mean of the series. A plot of the sample values of the autocorrelation against the lag is known as the autocorrelation function and is a basic tool in the analysis of time series particularly for indicating possibly suitable models for the series.

1.4 Summary of the present work

The following are the objectives of the study.

1. To study generalizations of Marshall-Olkin family of distributions such as Exponentiated Marshall-Olkin family, Negative Binomial Extreme Stable family and Harris family.
2. To develop generalizations of life distributions like, Exponential, Weibull, Fréchet, Lindley, Pareto, Rayleigh, Weibull etc and to study various properties of these distributions.
3. To obtain maximum likelihood estimates of the parameters using R program.
4. To develop new time series models and autoregressive minification processes.
5. To apply these distributions in various areas such as record values and reliability modeling.
6. To estimate reliability for stress-strength analysis using MATLAB.
7. To develop the acceptance sampling and reliability test plans with respect to some of these distributions.

The research concentrates on various generalizations of Marshall-Olkin family of distributions. The project report consists of 10 chapters. Chapter 1 covers an introduction to the topic of study as well as a brief review of literature and basic concepts. In chapter 2, a detailed study on the record value theory associated with Marshall-Olkin extended exponential distribution is conducted. Using the mean, variance and covariance of upper record values of the extended model BLUE's of location and scale parameters are obtained and future records are predicted which has a number of practical uses. The 95% confidence interval for location and scale parameters are also computed. MATLAB programs are developed for this purpose. The result is verified for a real data set. Entropy of record values

are derived. Stress-strength analysis is carried out and the validity of the estimate of reliability so obtained is studied through simulation studies. Parameter estimation of AR(1) minification process already developed is done and the result is verified for a simulated data. The auto covariance function and auto correlation function at lag 1 are obtained and graphs are drawn for the same.

Chapter 4 is devoted to the discussion of the newly developed distribution called Marshall-Olkin extended Fréchet distribution. A detailed study regarding the properties of probability density function and hazard rate function is carried out. Some of the properties of the distribution including *Renyi* entropy are derived. Some results based on order statistics are established. The new model is fitted to a real data set on the survival times of injected guinea pigs and verified to be a better fit compared to the Fréchet, the exponentiated Fréchet and beta Fréchet distributions by various statistical techniques.

Various applications of Marshall-Olkin extended Fréchet distribution are considered in chapter 4. Reliability of a system following Marshall-Olkin Fréchet distribution under stress-strength model is estimated and its validity is measured in terms of average bias and average mean square error calculated from the simulated N estimates of R . The average length of the asymptotic 95% confidence intervals and coverage probability, for the estimates obtained by simulation are evaluated. A reliability test plan is developed for products with life time following the new distribution. Minimum sample size required is determined to assure a minimum average life needed when the life test is terminated at a pre assigned time t such that the observed number of failures does not exceed a given acceptance number c . The operating characteristic values and the minimum value of the ratio of true average life to required average life for various sampling plans are tabulated. Four different minification processes are developed for the model and some of its properties are studied. Sample path properties are explored in all cases.

Chapter 5 discusses Marshall-Olkin Exponentiated Generalized Exponential Distribution and its Applications. Exponentiated generalized Exponential distribution and its prop-

erties are considered. Marshall-Olkin Exponentiated generalized Exponential Distribution and its properties are discussed. The quantiles and order statistics are considered. The maximum likelihood estimates are obtained and applied to a real data set on carbon fibers. Reliability of a system following Marshall-Olkin Exponentiated generalized Exponential distribution under stress-strength model is estimated and its validity is measured in terms of average bias and average mean square error calculated from the simulated N estimates. The average length of the asymptotic 95% confidence intervals and coverage probability for the estimates obtained by simulation are evaluated.

Chapter 6 concentrates on Marshall-Olkin Exponentiated Generalized Fréchet Distribution and its Applications. First we discuss the important properties of the Exponentiated generalized Fréchet distribution. Marshall-Olkin Exponentiated generalized Fréchet Distribution and its properties are discussed. We consider the quantiles and distribution of order statistics. The maximum likelihood estimates are obtained and applied to a real data set to compare the new distribution with Exponentiated generalized Fréchet distribution. Reliability of a system following Marshall-Olkin Exponentiated Generalized Fréchet distribution under stress-strength model is estimated. Its validity is examined using average bias and average mean square error calculated from the simulated values. Simulation studies are conducted to estimate the average length of the asymptotic 95% confidence intervals and coverage probability.

Chapter 7 presents Exponentiated Marshall-Olkin Exponential and Weibull distribution. The generalization introduced by Jayakumar and Thomas (2008) is applied here. Exponentiated Marshall-Olkin Exponential distribution and Exponentiated Marshall-Olkin Weibull distribution are considered. Various properties are studied including quantiles, order statistics, record values and Rényi entropy. Estimation of parameters is also considered. A real data set is analyzed as an application.

Chapter 8, introduces Negative Binomial Extreme Stable Marshall-Olkin Extended Lindley Distribution and its Applications. We first consider the properties of Extended

Lindley distribution. Negative binomial extreme stable Marshall-Olkin Extended Lindley Distribution and its properties are discussed. The expression of quantiles and the distribution of order statistics are derived. Record values associated with the new family is also considered. The maximum likelihood estimates of the distribution is obtained by using R programme and applied to a real data set on failure times of the air conditioning system of an airplane.

In chapter 9, a new distribution namely Negative Binomial Extreme Stable Marshall-Olkin Pareto distribution is introduced. The distributional properties are also considered. A reliability test plan is developed for products with lifetime following the new distribution. Minimum sample size required is determined to assure a minimum average life needed when the life test is terminated at a pre assigned time t such that the observed number of failures does not exceed a given acceptance number c . The operating characteristic values and the minimum value of the ratio of true average life to required average life for various sampling plans are tabulated.

Chapter 10 discusses Negative Binomial Marshall-Olkin Rayleigh Distribution and its Applications. Quantiles and order statistics of the distribution are obtained. We also develop the reliability test plan for the distribution. Minimum sample size required is determined to assure a minimum average life needed when the life test is terminated at a pre assigned time t such that the observed number of failures does not exceed a given acceptance number c . The operating characteristic values and the minimum value of the ratio of true average life to required average life for various sampling plans are tabulated.

Publications

1. Jose, K.K., Remya Sivadas (2014). Marshall-Olkin Exponentiated Generalized Fréchet Distribution and its Applications. *Journal of Probability and Statistical Sciences*. (in press)
2. Jose, K.K., Remya Sivadas (2014). Negative Binomial Extreme Stable Marshall-Olkin Extended Lindley Distribution and its Applications. *International Journal of*

Statistics and Probability. (Accepted for publication)

3. Krishna,E.,Jose,K.K., Alice, T., Miroslav M.Ristic.(2013).Marshall-Olkin Fréchet distribution',*Communications in Statistics: Theory and Methods*. (Accepted for publication).
4. Krishna,E.,Jose,K.K.,Miroslav M.Ristic.,(2013.)Applications of Marshall-Olkin Fréchet distribution, *Communications in Statistics: Computations and Simulations* (Accepted for publication).
5. Jose,K.K., Krishna,E., Miroslav M.Ristic.(2014).Marshall-Olkin extended exponential distribution and its applications in record value theory, stress-strength analysis and time series modeling. *Journal of Applied Statistical Science* (Accepted for publication).
6. Jose, K.K., Remya Sivadas (2014). Negative Binomial Marshall- Olkin Rayleigh Distribution and its Applications. *Journal of Economic Quality Control*.(under revision)
7. Jose, K.K., Remya Sivadas (2015). Generalized Marshall-Olkin Exponential and Weibull Distributions. *Journal of Reliability and Statistical Science*.(under revision)
8. Jose, K.K., Remya Sivadas (2014). Marshall-Olkin Exponentiated Generalized Exponential Distribution and its Applications. *Journal of Data Science*.(submitted)
9. Jose, K.K., Remya Sivadas (2014). Reliability Test Plan for Negative Binomial Extreme Stable Marshall- Olkin Pareto Distribution. *Journal of Applied Statistical Sciences*.(submitted)

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CHAPTER 2

On Marshall- Olkin Extended Exponential Distribution and Applications

2.1 Introduction

Exponential distributions play a central role in analysis of lifetime or survival data, in part because of their convenient statistical theory, their important lack of memory property and their constant hazard rates. In circumstances where the one-parameter family of exponential distributions is not sufficiently broad, a number of wider families such as the gamma, Weibull and Gompertz-Makeham distributions are in common use; these families and their usefulness are described by various authors.(see Johnson et al 2004).

By various methods, new parameters can be introduced to expand families of distributions for added flexibility or to construct covariate models. Introduction of a scale parameter leads to the accelerated life model, and taking powers of the survival function introduces a parameter that leads to the proportional hazards model. For instance, the

family of Weibull distributions contains the exponential distributions and is constructed by taking powers of exponentially distributed random variables. The family of gamma distributions also contains the exponential distributions, and is constructed by taking powers of the Laplace transform.

In this chapter, we introduce the Marshall-Olkin Extended Exponential distribution $MOEE(\alpha, \lambda)$ and its properties are studied. we discuss $MOEE(\alpha)$ distributions with special emphasis on record value theory. We derive the entropy of record value distribution and entropy is calculated for various record values. We also obtain an estimate of reliability in the context of stress strength analysis and average bias, average mean square error, average confidence interval and coverage probability for the estimate is tabulated numerically for a simulated data .Finally we introduce first order stationary autoregressive processes with exponential marginals and the sample path properties are explored. The probability p is estimated and the standard error of the estimated value is calculated numerically by simulation.

2.2 Marshall -Olkin Extended Exponential Distribution

When $\bar{F}(x) = e^{-\sigma x}$, $x \geq 0$ is the survival function of exponential distribution, then by(??) we have the Marshall -Olkin Extended Exponential $MOEE(\alpha, \sigma)$ distribution with survival function,

$$\bar{G}(x) = \frac{\alpha}{e^{\sigma x} - \bar{\alpha}} \quad x \geq 0, \quad \sigma > 0, \quad \alpha > 0, \quad \bar{\alpha} = 1 - \alpha \quad (2.2.1)$$

Then the p.d.f. is

$$g(x) = \frac{\alpha \sigma e^{\sigma x}}{[e^{\sigma x} - \bar{\alpha}]^2} \quad x \geq 0, \quad \sigma > 0, \quad \alpha > 0, \quad \bar{\alpha} = 1 - \alpha. \quad (2.2.2)$$

Direct evaluation shows that,

$$E(X) = -\frac{\alpha \log \alpha}{\sigma \bar{\alpha}}$$

The hazard rate is

$$h(x) = \frac{\sigma e^{\sigma x}}{e^{\sigma x} - \bar{\alpha}} \quad x \geq 0, \quad \alpha > 0. \quad (2.2.3)$$

The graph of $h(x)$ is drawn in Figure 2.1. It can be seen that the hazard rate is DFR for $\alpha < 1$, and IFR for $\alpha > 1$. Note that for $\alpha = 1$ $h(x)=1$, showing constant failure rate. This establishes the wide applicability of the MOEE distribution in reliability modeling.

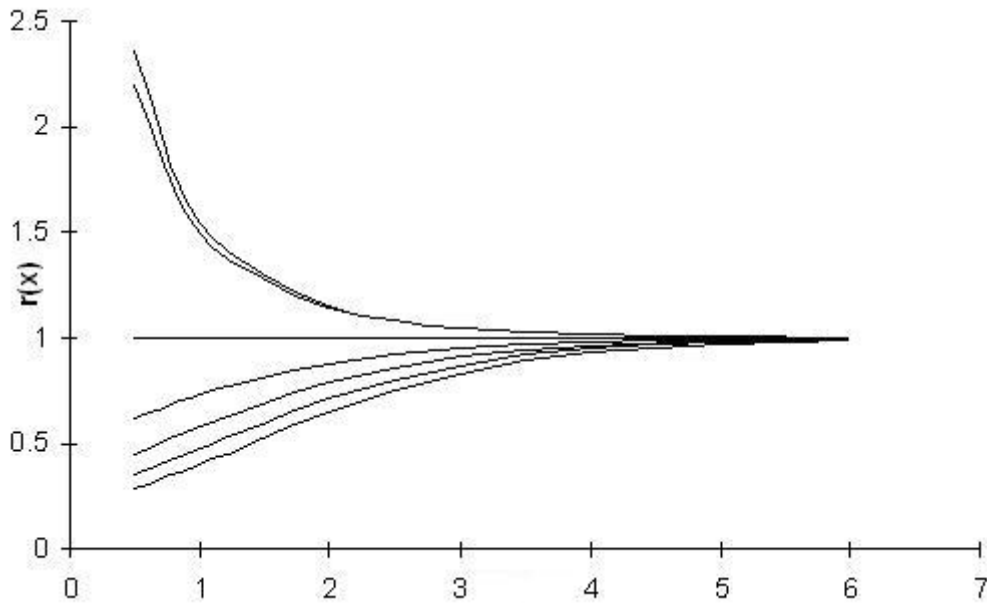


Figure 2.1: Hazard rate function of MOEE (α, σ) for various values of α and σ

2.3 Record Value Theory for Marshall-Olkin Extended Exponential Distribution

Chandler(1952)introduced the study of record values and documented many of the basic properties of records.Nagaraja(1988),Nevzorov(1988,2001),Arnold and Balakrishnan (1989), Balakrishnan and Ahsanullah (1994), Ahsanullah (1995,1998), Sultan et al. (2003), etc. have made significant contributions to the theory of records. Arnold et al. (1998) provide an excellent discussion on various results with respect to record values. Now we

derive some record statistics with respect to Marshall-Olkin Extended Exponential distribution with $\sigma = 1$ for which the p.d.f is

$$g(x) = \frac{\alpha e^x}{(e^x - \bar{\alpha})^2}, x > 0, \alpha > 0, \bar{\alpha} = 1 - \bar{\alpha} \quad (2.3.1)$$

By (1.3.1) the density function of the n^{th} record for $MOEE(\alpha)$ distribution is given by

$$g_{R_n}(x) = \frac{\alpha e^x}{(n-1)![e^x - (1-\alpha)]^2} \left[-\ln \left(\frac{\alpha}{e^x - (1-\alpha)} \right) \right]^{n-1}, \quad 0 < x < \infty \quad (2.3.2)$$

Then the single moment of n^{th} record statistic can be written as

$$\beta_n = \int_0^\infty \ln(\bar{\alpha} + \alpha e^u) \frac{u^{n-1}}{(n-1)!} e^{-u} du \quad (2.3.3)$$

Theorem 2.3.1. *The single moment of n th upper record value for $\alpha > 0.5$ is given by*

$$\beta_n = \ln(\alpha) + n - \sum_{i=1}^{\infty} \frac{k^i}{i(i+1)^n}, \quad \text{where } k = 1 - \frac{1}{\alpha}. \quad (2.3.4)$$

and consequently, for $n \geq 2$

$$\beta_n = \beta_{n-1} + \sum_{i=0}^{\infty} \frac{k^i}{(i+1)^n} \quad (2.3.5)$$

Proof From (2.3.3) and using the fact that $\ln[1 - ke^{-u}] = -\sum_{i=1}^{\infty} \frac{k^i e^{-iu}}{i}$

$$\beta_n = \ln(\alpha) \int_0^\infty \frac{u^{n-1} e^{-u}}{(n-1)!} du + \int_0^\infty \frac{u^n e^{-u}}{(n-1)!} du - \sum_{i=1}^{\infty} \frac{k^i}{i} \int_0^\infty \frac{e^{-(i+1)u} u^{n-1}}{(n-1)!} du$$

which on evaluation directly gives (2.3.4)

Table 2.1: Mean of upper record values

n	$\alpha = 1$	$\alpha = 1.5$	$\alpha = 2$	$\alpha = 2.5$	$\alpha = 3$	$\alpha = 3.5$	$\alpha = 4$
1	1	1.2164	1.3863	1.5272	1.6479	1.7539	1.8484
2	2	2.3150	2.5508	2.7398	2.8978	3.0337	3.1530
3	3	3.3615	3.6252	3.8331	4.0049	4.1513	4.2789
4	4	4.3839	4.6602	4.8762	5.0537	5.2043	5.3352
5	5	5.3948	5.677	5.8967	6.0767	6.2292	6.3615
6	6	6.4002	6.6852	6.9066	7.0879	7.2412	7.3741
7	7	7.4028	7.6892	7.9115	8.0933	8.2471	8.3803

Now

$$\beta_n = \ln(\alpha) + n - \sum_{i=1}^{\infty} \frac{k^i}{(i+1)^{n-1}} \left[\frac{1}{i} - \frac{1}{i+1} \right]$$

simplifying we get the recurrence relation (2.3.5)

Using the result(2.3.4) the mean of record values from $MOEE(\alpha)$ for $\alpha = 1.0(0.5)4.0$ are evaluated and presented in table 2.1

Theorem 2.3.2. *The second single moment of n^{th} upper record value is*

$$\begin{aligned} \beta_n^2 &= \ln(\alpha)^2 + n(n+1 + 2\ln(\alpha)) - 2n \sum_{i=1}^{\infty} \frac{k^i}{i(i+1)^{n+1}} - 2\ln(\alpha) \\ &\times \sum_{i=1}^{\infty} \frac{k^i}{i(i+1)^n} + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{k^{i+j}}{ij(i+j+1)^n} \end{aligned} \quad (2.3.6)$$

Proof From (2.3.3) the 2^{nd} single moment of n^{th} record value is given by

$$\begin{aligned} \beta_n^2 &= \int_0^{\infty} \{ \ln[\alpha e^u(1 - ke^{-u})] \}^2 \frac{u^{n-1} e^{-u}}{(n-1)!} du, \quad k = 1 - \frac{1}{\alpha} \\ &= (\ln \alpha)^2 + n(n+1) + 2n \ln \alpha - 2n \sum_{i=1}^{\infty} \frac{k^i}{i(i+1)^{(n+1)}} - 2 \\ &\times \ln \alpha \sum_{i=1}^{\infty} \frac{k^i}{i(i+1)^n} + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{k^{i+j}}{ij} \int_0^{\infty} e^{-(i+j+1)u} \frac{u^{n-1}}{(n-1)!} du \end{aligned}$$

on simplification using the fact that $(a_1 + a_2)^2 = \sum_{i=1}^2 \sum_{j=1}^2 a_i a_j$ we get (2.3.6).

By (1.3.2) the joint p.d.f. of m^{th} and n^{th} record values of MOEE (α) distribution is given by

$$g_{R_m, R_n}(x) = \frac{\alpha^2 \left[-\ln \left\{ \frac{\alpha}{e^x - (1 - \alpha)} \right\} \right]^{m-1}}{(m-1)!} \frac{1}{[e^x - (1 - \alpha)]} \\ \times \frac{\left[-\ln \left\{ \frac{e^x - (1 - \alpha)}{e^y - (1 - \alpha)} \right\} \right]^{n-m-1}}{(n-m-1)!} \\ \times \frac{e^y}{[e^y - (1 - \alpha)]^2}, 0 < x < y < \infty$$

Theorem 2.3.3. For $1 \leq m \leq n$ the product moment

$$\beta_{m,n} = (\ln \alpha)^2 + \ln \alpha(m+n) + m(n+1) - [\ln \alpha + (n-m)] \\ \times \sum_{i=1}^{\infty} \frac{k^i}{i(i+1)^m} - m \sum_{i=1}^{\infty} \frac{k^i}{i(i+1)^{m+1}} - \ln \alpha \sum_{i=1}^{\infty} \frac{k^j}{j(j+1)^n} - m \\ \times \sum_{j=1}^{\infty} \frac{k^j}{j(j+1)^{n+1}} + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{k^{(i+j)}}{ij(j+1)^{n-m}(i+j+1)^m} \quad (2.3.7)$$

Proof:

$$\beta_{m,n} = \frac{\alpha}{(m-1)!} \int_0^{\infty} x \left[-\ln \left(\frac{\alpha}{e^x - \bar{\alpha}} \right) \right]^{m-1} \frac{e^x}{e^x - \bar{\alpha}} I_x dx \quad (2.3.8)$$

where

$$I_x = \frac{1}{(n-m-1)!} \int_x^{\infty} \frac{ye^y}{(e^y - \bar{\alpha})^2} \left[-\ln \left(\frac{e^x - \bar{\alpha}}{e^y - \bar{\alpha}} \right) \right]^{(n-m-1)} dy$$

now making use of the transformation $u = -\ln \left(\frac{e^x - \bar{\alpha}}{e^y - \bar{\alpha}} \right)$

and writing $\ln \left[1 - \left(\frac{\alpha-1}{e^x - \bar{\alpha}} \right) e^{-u} \right] = -\sum_{i=1}^{\infty} \left(\frac{\alpha-1}{e^x - \bar{\alpha}} \right)^i \frac{e^{-iu}}{i}$ we get

$$I_x = \frac{1}{(e^x - \bar{\alpha})} \left[\ln(e^x - \bar{\alpha}) + (n-m) - \sum_{i=1}^{\infty} \left(\frac{\alpha-1}{e^x - \bar{\alpha}} \right)^i \frac{1}{i(i+1)^{n-m}} \right]$$

Table 2.2: Variance and covariance of upper record values

m	n	$\alpha = 1$	$\alpha = 1.5$	$\alpha = 2$	$\alpha = 2.5$	$\alpha = 3$	$\alpha = 3.5$	$\alpha = 4$
1	1	1	1.2109	1.3681	1.4924	1.5946	1.6809	1.7551
	2	1	1.1515	1.2563	1.3348	1.3965	1.4466	1.4885
	3	1	1.1241	1.2075	1.2688	1.3164	1.3546	1.3863
	4	1	1.111	1.1851	1.2393	1.2812	1.3148	1.3426
	5	1	1.1047	1.1745	1.2256	1.265	1.2966	1.3228
	6	1	1.1017	1.1694	1.219	1.2573	1.288	1.3135
	7	1	1.1001	1.1669	1.2158	1.2535	1.2839	1.309
2	2	2	2.193	2.3164	2.4035	2.4689	2.52	2.5613
	3	2	2.1411	2.2274	2.2865	2.3297	2.3629	2.3893
	4	2	2.1164	2.1866	2.2341	2.2686	2.295	2.3158
	5	2	2.1045	2.1672	2.2096	2.2404	2.2639	2.2824
	6	2	2.0987	2.1579	2.1979	2.227	2.2492	2.2667
	7	2	2.0958	2.1533	2.1922	2.2205	2.2421	2.2591
	3	3	3	3.1371	3.217	3.2697	3.3074	3.3356
4		3	3.1013	3.1587	3.196	3.2222	3.2418	3.2569
5		3	3.0839	3.1311	3.1616	3.1829	3.1988	3.211
6		3	3.0754	3.1178	3.1451	3.1642	3.1784	3.1894
7		3	3.0712	3.1113	3.1371	3.1552	3.1686	3.179
4	4	4	4.0884	4.137	4.1679	4.1894	4.2051	4.2172
	5	4	4.0657	4.1012	4.1236	4.139	4.1503	4.1589
	6	4	4.0545	4.0839	4.1024	4.115	4.1243	4.1314
	7	4	4.0491	4.0755	4.092	4.1034	4.1117	4.1181
5	5	5	5.0541	5.0828	5.1007	5.1129	5.1217	5.1285
	6	5	5.0403	5.0615	5.0747	5.0836	5.09	5.0949
	7	5	5.0335	5.0511	5.062	5.0694	5.0747	5.0787
6	6	6	6.032	6.0487	6.0589	6.0658	6.0708	6.0745
	7	6	6.0239	6.0363	6.0438	6.0489	6.0526	6.0554
7	7	7	7.0185	7.028	7.0338	7.0377	7.0405	7.0426

substituting the expression of I_x in (2.3.8) and using the transformation $t = -\ln\left(\frac{\alpha}{e^x - \alpha}\right)$ yields (2.3.7). Using (2.3.4),(2.3.6)and (2.3.7)numerical values of variance and covariance of upper record values are obtained by Matlab program for various values of $\alpha = 1(0.5)4$ and is presented in table 2.2.

2.3.1 Estimation of the location and scale parameters

In industry experiments the number of measurements can be made lesser if the record values are observed instead of complete sample for estimation of parameters. There are also situations in which an observation is stored if it is a record value. This includes studies in meteorology, hydrology, seismology athletic events and mining. Recently much stud-

ies have been made on parametric and non parametric inferences based on record values. Sultan et al(2000), Raqab (2002) obtained inference for generalised exponential distribution based on record statistics. Soliman et al(2006) made a comparative study on Bayesian and non-Bayesian estimates using record statistics from Weibull model. Sultan et al(2008) obtained the estimation from record values and predicted future records for gamma distribution. Sultan(2010) discussed different methods of estimation based on record values from inverse Weibull distribution.

Consider the general location-scale family of distributions with cdf $F(x, \mu, \sigma) = F(\frac{x-\mu}{\sigma})$ and pdf $f(x, \mu, \sigma) = \frac{1}{\sigma}f(\frac{x-\mu}{\sigma})$ and assume that the upper record values R_1, R_2, \dots, R_n are available. Then BLUE's of μ and σ are given respectively by, (see Balakrishnan and Cochen, 1991)

$$\mu^* = \frac{\beta^T \Sigma^{-1} \beta \mathbf{1}^T \Sigma^{-1} - \beta^T \Sigma^{-1} \mathbf{1} \beta^T \Sigma^{-1}}{(\beta^T \Sigma^{-1} \beta)(\mathbf{1}^T \Sigma^{-1} \mathbf{1}) - (\beta^T \Sigma^{-1} \mathbf{1})^2} \mathbf{R} = \sum_{i=1}^n a_i R_i \quad (2.3.9)$$

$$\sigma^* = \frac{\mathbf{1}^T \Sigma^{-1} \mathbf{1} \beta^T \Sigma^{-1} - \mathbf{1}^T \Sigma^{-1} \beta \mathbf{1}^T \Sigma^{-1}}{(\beta^T \Sigma^{-1} \beta)(\mathbf{1}^T \Sigma^{-1} \mathbf{1}) - (\beta^T \Sigma^{-1} \mathbf{1})^2} \mathbf{R} = \sum_{i=1}^n b_i R_i \quad (2.3.10)$$

where β denotes the column vector of the expected values of observed upper record values from the distribution $F(x)$, Σ denotes the variance-covariance matrix of the record values from the distribution $F(x)$, and $\mathbf{1}$ is a column vector of dimension n with all its entries as 1.

The three parameter Marshall-Olkin extended exponential distribution has the probability density function given by

$$g(y) = \frac{\alpha e^{-\frac{(y-\mu)}{\sigma}}}{\sigma (e^{-\frac{(y-\mu)}{\sigma}} - \bar{\alpha})^2}, y > \mu, \alpha, \sigma > 0,$$

where α, μ and σ are the shape, location and scale parameters respectively. Let $x = \frac{y-\mu}{\sigma}$. Then X has one parameter MOEE density function as given in (2.3.1).

By making use of means, variances and covariances presented in Table 2.1 and 2.2 we

calculate the coefficients of **BLUEs** a_i and b_i , $i=1,2,\dots,n$ for different values of shape parameter α and n and presented in Table 2.3 and Table 2.4. It can be noted from these tables that $\sum_{i=1}^n a_i = 1$ and $\sum_{i=1}^n b_i = 0$

The variances and covariance of the BLUE's of μ and σ are given by(see Balakrishnan and Cochen,1991)

$$Var(\mu^*) = \sigma^2 \left\{ \frac{\beta^T \Sigma^{-1} \beta}{(\beta^T \Sigma^{-1} \beta)(\mathbf{1}^T \Sigma^{-1} \mathbf{1}) - (\beta^T \Sigma^{-1} \mathbf{1})^2} \right\} = \sigma^2 V_1$$

$$Var(\sigma^*) = \sigma^2 \left\{ \frac{\mathbf{1}^T \Sigma^{-1} \mathbf{1}}{(\beta^T \Sigma^{-1} \beta)(\mathbf{1}^T \Sigma^{-1} \mathbf{1}) - (\beta^T \Sigma^{-1} \mathbf{1})^2} \right\} = \sigma^2 V_2$$

$$Cov(\mu^*, \sigma^*) = \sigma^2 \left\{ \frac{-\beta^T \Sigma^{-1} \mathbf{1}}{(\beta^T \Sigma^{-1} \beta)(\mathbf{1}^T \Sigma^{-1} \mathbf{1}) - (\beta^T \Sigma^{-1} \mathbf{1})^2} \right\} = \sigma^2 V_3$$

Using these results Variance and covariances of the BLUE's of μ and σ can be obtained in terms of σ^2 and is presented in Table 2.5

Example:Consider a simulated data of failure times which follow MOEE distribution with $\alpha = 1.5$,

9.9936,9.0684,10.0644,13.3159,9.6027,13.93718,8.3373,8.8017,9.9252,10.6296.

The observed upper record values are then,

9.9936,10.0644,13.3159,13.93718.

With $n=4$, $\alpha = 1.5$,the BLUE's of μ and σ cab be computed using(2.3.9),(2.3.10)and Tables 2.3 and 2.4.The estimates are

$$\mu^* = 8.5542 \text{ and } \sigma^* = 1.248$$

The corresponding variances and covariance of μ^* and σ^* can be obtained from Table 2.5.

$$Var(\mu^*) = 1.7046, Var(\sigma^*) = 0.3067 \text{ and } Cov(\mu^* \sigma^*) = -0.4047$$

Table 2.3: Coefficients of the BLUE of μ

n	$\alpha = 1$	$\alpha = 1.5$	$\alpha = 2$	$\alpha = 2.5$	$\alpha = 3$	$\alpha = 3.5$	$\alpha = 4$
2	2.0000 -1.0000	2.1072 -1.1072	2.1905 -1.1905	2.2594 -1.2594	2.3184 -1.3184	2.3704 -1.3704	2.4168 -1.4168
3	1.5000 0.0000 -0.5000	1.5533 0.0282 -0.5815	1.5943 0.0518 -0.6461	1.6281 0.0721 -0.7002	1.6568 0.0901 -0.7469	1.6819 0.1066 -0.7885	1.7043 0.1212 -0.8256
4	1.3333 0.0000 0.0000 -0.3333	1.3628 0.0279 0.0093 -0.4000	1.3852 0.0509 0.0172 -0.4533	1.4037 0.0704 0.0239 -0.4979	1.4194 0.0875 0.0294 -0.5363	1.4330 0.1030 0.0344 -0.5705	1.4454 0.1166 0.0388 -0.6008
5	1.2500 0.0000 0.0000 0.0000 -0.2500	1.2647 0.0278 0.0104 0.0035 -0.3064	1.2758 0.0503 0.0191 0.0061 -0.3514	1.2851 0.0693 0.0264 0.0084 -0.3891	1.2929 0.0859 0.0323 0.0102 -0.4214	1.2998 0.1008 0.0377 0.0117 -0.4500	1.3061 0.1138 0.0423 0.0131 -0.4754
6	1.2000 0.0000 0.0000 0.0000 0.0000 -0.2000	1.1705 0.0577 0.0245 0.0096 0.0035 -0.2657	1.2077 0.0500 0.0202 0.0073 0.0023 -0.2875	1.2106 0.0686 0.0278 0.0100 0.0030 -0.3199	1.2130 0.0849 0.0341 0.0121 0.0037 -0.3478	1.2153 0.0993 0.0396 0.0139 0.0041 -0.3722	1.2176 0.1119 0.0444 0.0155 0.0045 -0.3940
7	1.1667 0.0000 0.0000 0.0000 0.0000 0.0000 -0.1667	1.1635 0.0276 0.0115 0.0047 0.0018 0.0003 -0.2094	1.1609 0.0497 0.0210 0.0081 0.0029 0.0009 -0.2435	1.1590 0.0681 0.0288 0.0110 0.0038 0.0012 -0.2719	1.1576 0.0841 0.0353 0.0133 0.0047 0.0012 -0.2961	1.1564 0.0982 0.0409 0.0153 0.0052 0.0016 -0.3176	1.1558 0.1105 0.0459 0.0171 0.0057 0.0016 -0.3365

Table 2.4: Coefficients for the BLUE of σ

n	$\alpha = 1$	$\alpha = 1.5$	$\alpha = 2$	$\alpha = 2.5$	$\alpha = 3$	$\alpha = 3.5$	$\alpha = 4$
2	-1.0000	-0.9102	-0.8587	-0.8247	-0.8001	-0.7814	-0.7665
	1.0000	0.9102	0.8587	0.8247	0.8001	0.7814	0.7665
3	-0.5000	-0.4658	-0.4455	-0.4318	-0.4217	-0.4138	-0.4075
	0.0000	-0.0008	-0.0023	-0.0039	-0.0055	-0.0071	-0.0085
	0.5000	0.4666	0.4478	0.4357	0.4272	0.4209	0.4160
4	-0.3333	-0.3164	-0.3059	-0.2986	-0.2930	-0.2885	-0.2848
	0.0000	-0.0006	-0.0017	-0.0029	-0.0041	-0.0053	-0.0063
	0.0000	0.0034	0.0050	0.0058	0.0063	0.0064	0.0065
	0.3333	0.3136	0.3026	0.2957	0.2908	0.2873	0.2846
5	-0.2500	-0.2409	-0.2348	-0.2304	-0.2269	-0.2239	-0.2215
	0.0000	-0.0005	-0.0013	-0.0023	-0.0033	-0.0042	-0.0051
	0.0000	0.0026	0.0038	0.0044	0.0047	0.0049	0.0050
	0.0000	0.0026	0.0039	0.0047	0.0051	0.0054	0.0056
	0.2500	0.2361	0.2284	0.2236	0.2203	0.2179	0.2160
6	-0.2000	-0.1890	-0.1913	-0.1884	-0.1860	-0.1840	-0.1823
	0.0000	-0.0067	-0.0011	-0.0019	-0.0027	-0.0035	-0.0042
	0.0000	0.0004	0.0030	0.0036	0.0038	0.0040	0.0040
	0.0000	0.0020	0.0032	0.0038	0.0042	0.0044	0.0045
	0.0000	0.0018	0.0024	0.0028	0.0030	0.0033	0.0034
	0.2000	0.1915	0.1838	0.1802	0.1777	0.1759	0.1746
7	-0.1667	-0.1640	-0.1617	-0.1598	-0.1581	-0.1566	-0.1553
	0.0000	-0.0003	-0.0009	-0.0016	-0.0023	-0.0030	-0.0036
	0.0000	0.0018	0.0026	0.0030	0.0032	0.0034	0.0034
	0.0000	0.0018	0.0027	0.0032	0.0035	0.0037	0.0039
	0.0000	0.0013	0.0020	0.0024	0.0026	0.0027	0.0029
	0.0000	0.0009	0.0013	0.0015	0.0017	0.0018	0.0018
	0.1667	0.1585	0.1541	0.1513	0.1493	0.1480	0.1470

Table 2.5: Variance and covariances of the BLUE's of μ and σ in terms of σ^2

n	$\alpha = 1$	$\alpha = 1.5$	$\alpha = 2$	$\alpha = 2.5$	$\alpha = 3$	$\alpha = 3.5$	$\alpha = 4$
2	2.0000 1.0000 -1.0000	2.6921 0.9122 -1.1636	3.2951 0.8642 -1.2940	3.8345 0.8340 -1.4036	4.3254 0.8133 -1.4986	4.7791 0.7984 -1.5834	5.1993 0.7870 -1.6590
3	1.5000 0.5000 -0.5000	1.9841 0.4563 -0.5955	2.3974 0.4329 -0.6718	2.7612 0.4184 -0.7357	3.0878 0.4084 -0.7908	3.3860 0.4014 -0.8397	3.6600 0.3961 -0.8833
4	1.3333 0.3333 -0.3333	1.7406 0.3067 -0.4047	2.0823 0.2925 -0.4614	2.3792 0.2837 -0.5089	2.6430 0.2777 -0.5497	2.8817 0.2735 -0.5857	3.0995 0.2703 -0.6178
5	1.2500 0.2500 -0.2500	1.6153 0.2323 -0.3081	1.9174 0.2228 -0.3542	2.1770 0.2169 -0.3927	2.4059 0.2129 -0.4258	2.6113 0.2101 -0.4548	2.7977 0.2080 -0.4807
6	1.2000 0.2000 -0.2000	1.5382 0.1875 -0.2493	1.8147 0.1808 -0.2886	2.0501 0.1766 -0.3212	2.2560 0.1738 -0.3492	2.4391 0.1716 -0.3734	2.6057 0.1703 -0.3956
7	1.1667 0.1667 -0.1667	1.4859 0.1575 -0.2097	1.7441 0.1525 -0.2439	1.9622 0.1494 -0.2723	2.1520 0.1473 -0.2967	2.3203 0.1458 -0.3181	2.4716 0.1447 -0.3370

2.4 Confidence interval

Through the pivotal quantities

$$R_1 = \frac{\mu^* - \mu}{\sigma\sqrt{V_1}}, \quad R_2 = \frac{\sigma^* - \sigma}{\sigma\sqrt{V_2}} \quad \text{and} \quad R_3 = \frac{\mu^* - \mu}{\sigma^*\sqrt{V_1}}$$

where μ^* and σ^* are the BLUE's of μ and σ we construct confidence interval for the location and scale parameters . We use R_1 and R_3 to construct CIs for μ when σ is known and when σ is unknown respectively while R_3 is used to construct CI's for σ . The construction of CI's require the percentage points of R_1, R_2 and R_3 which is obtained by using the BLUE's μ^* and σ^* via Monte carlo simulation based on 10000 runs and are presented in Table 2.6,2.7 and 2.8 respectively.

2.5 Application

Now we apply the inference procedure discussed in the previous section to upper records of simulated data sets of size $n=3,4,5,6$ and 7 (with $\mu = 0, \sigma = 1$ and $\alpha = 1.5$).The BLUE's are calculated using tables 2.3 and 2.4 and is presented in table 2.9 Using the BLUE's given in table 2.9 and the percentage points of R_1 and R_3 we construct 95% confidence interval for μ when σ known and σ unknown respectively through the formulae,

$$P(\mu^* - \sigma\sqrt{V_1}R_1(97.5) \leq \mu \leq \mu^* - \sigma\sqrt{V_1}R_1(2.5)) = 95\%$$

$$P(\mu^* - \sigma^*\sqrt{V_1}R_3(97.5) \leq \mu \leq \mu^* - \sigma^*\sqrt{V_1}R_3(2.5)) = 95\%$$

We also construct confidence interval for σ using percentage points of R_2 through the formula

$$P\left(\frac{\sigma^*}{1 + \sqrt{V_2}R_2(97.5)} \leq \sigma \leq \frac{\sigma^*}{1 + \sqrt{V_2}R_2(2.5)}\right) = 95\%$$

The result is presented in table 2.10

Table 2.6: Simulated percentage points of R_1

α	n	0.5%	2.5%	5%	95%	97.5%	99.5%
1	3	-2.8296	-2.1592	-1.859	1.5535	1.8638	2.4919
	4	-2.036	-1.5646	-1.3589	1.1329	1.4127	1.9046
	5	-1.6618	-1.2495	-1.0814	0.8481	1.0886	1.5583
	6	-1.3162	-1.0171	-0.8835	0.633	0.8488	1.2872
	7	-1.1594	-0.8969	-0.7726	0.5081	0.6791	1.0577
1.5	3	-2.7469	-2.1267	-1.8256	1.3433	1.6125	2.0987
	4	-2.0792	-1.568	-1.3435	0.9755	1.2015	1.6767
	5	-1.7621	-1.3661	-1.1965	0.9028	1.1398	1.6211
	6	-1.5433	-1.2122	-1.0447	0.6522	0.8659	1.2275
	7	-1.2433	-0.9933	-0.8666	0.5839	0.7746	1.1191
2	3	-2.9372	-2.2804	-1.9577	1.5304	1.7727	2.2355
	4	-2.1547	-1.6662	-1.4463	1.001	1.2494	1.6742
	5	-1.8518	-1.413	-1.2268	0.9311	1.1606	1.5419
	6	-1.6084	-1.2163	-1.0591	0.7614	0.9621	1.3124
	7	-1.386	-1.0601	-0.9331	0.6092	0.7899	1.1412
2.5	3	-3.0444	-2.2943	-1.978	1.5021	1.7637	2.2164
	4	-2.18	-1.6771	-1.4607	1.0432	1.262	1.7578
	5	-1.9563	-1.5005	-1.3072	0.9578	1.1741	1.5659
	6	-1.659	-1.2994	-1.127	0.773	0.9675	1.3537
	7	-1.4385	-1.1016	-0.9666	0.6717	0.8683	1.2118
3	3	-3.0072	-2.3132	-2.0032	1.5444	1.7876	2.2442
	4	-2.329	-1.7043	-1.4957	1.0514	1.2699	1.7056
	5	-1.9276	-1.5085	-1.3244	0.9799	1.2236	1.6289
	6	-1.6967	-1.3407	-1.1601	0.8103	1.0023	1.3681
	7	-1.5773	-1.1836	-1.0341	0.6741	0.8618	1.1855
3.5	3	-2.981	-2.3629	-2.0257	1.491	1.7473	2.1886
	4	-2.4141	-1.7946	-1.5459	1.0794	1.2766	1.6736
	5	-2.0119	-1.5752	-1.3675	1.0136	1.2454	1.6709
	6	-1.7613	-1.3636	-1.1757	0.8331	1.0474	1.4157
	7	-1.5573	-1.1891	-1.0476	0.6974	0.8853	1.2349
4	3	-3.1703	-2.4131	-2.0758	1.4774	1.7188	2.1222
	4	-2.4117	-1.8432	-1.587	1.0576	1.2802	1.6664
	5	-2.0393	-1.6271	-1.4089	0.9825	1.1939	1.6094
	6	-1.779	-1.3837	-1.2174	0.8429	1.0452	1.4303
	7	-1.5883	-1.2268	-1.081	0.7254	0.9267	1.2924

Table 2.7: Simulated percentage points of R_2

α	n	0.5%	2.5%	5%	95%	97.5%	99.5%
1	3	-1.2506	-1.0026	-0.8327	2.453	2.9989	4.0999
	4	-1.3996	-1.1541	-0.9928	1.5656	1.9704	3.0443
	5	-1.541	-1.3188	-1.1942	0.9047	1.3226	2.1704
	6	-1.733	-1.556	-1.4405	0.403	0.7204	1.4378
	7	-1.916	-1.7497	-1.6572	0.0506	0.3645	1.0337
1.5	3	-1.3247	-1.0896	-0.9345	2.2299	2.7888	3.8331
	4	-1.4715	-1.2285	-1.0846	1.3654	1.7744	2.714
	5	-1.6068	-1.3679	-1.2072	0.9354	1.2378	2.0441
	6	-1.7535	-1.5495	-1.4251	0.462	0.752	1.3985
	7	-1.9489	-1.7676	-1.6626	0.0409	0.3244	0.936
2	3	-1.3364	-1.0862	-0.9049	2.4359	2.9387	4.0115
	4	-1.5098	-1.2669	-1.1093	1.4517	1.8383	2.8977
	5	-1.6471	-1.4	-1.258	0.9388	1.2818	2.0531
	6	-1.8033	-1.5679	-1.4521	0.4505	0.777	1.4362
	7	-1.9476	-1.7578	-1.6385	0.0444	0.3369	1.005
2.5	3	-1.3546	-1.1203	-0.9356	2.4171	2.9537	4.1176
	4	-1.5525	-1.3032	-1.1375	1.4003	1.7763	2.6919
	5	-1.6252	-1.391	-1.2378	1.0057	1.3852	2.133
	6	-1.7704	-1.5579	-1.4247	0.5361	0.8411	1.4941
	7	-1.9689	-1.7756	-1.6575	0.0504	0.3189	0.9403
3	3	-1.378	-1.1623	-0.9745	2.4274	2.8905	3.8861
	4	-1.5728	-1.3023	-1.1544	1.3693	1.7548	2.6961
	5	-1.6821	-1.4096	-1.252	0.992	1.3542	2.1185
	6	-1.8553	-1.5797	-1.4424	0.545	0.8509	1.6086
	7	-1.9962	-1.7909	-1.6659	0.1346	0.414	1.073
3.5	3	-1.3688	-1.1422	-0.9844	2.4497	2.8813	3.9018
	4	-1.5727	-1.3152	-1.1552	1.4768	1.8687	2.8386
	5	-1.7154	-1.4377	-1.2764	1.0217	1.3664	2.1111
	6	-1.8318	-1.582	-1.4572	0.5079	0.7956	1.4977
	7	-1.9918	-1.7892	-1.6604	0.1175	0.351	1.0937
4	3	-1.4079	-1.161	-0.9921	2.4939	3.0147	4.0791
	4	-1.5692	-1.324	-1.1664	1.5152	1.929	2.8365
	5	-1.6899	-1.425	-1.2633	1.0731	1.419	2.2244
	6	-1.8259	-1.6027	-1.4583	0.5409	0.8475	1.563
	7	-2.0118	-1.7927	-1.6489	0.123	0.3674	1.1029

Table 2.8: Simulated percentage points of R_3

α	n	0.5%	2.5%	5%	95%	97.5%	99.5%
1	3	-0.8123	-0.7976	-0.7773	3.5099	5.6257	13.8717
	4	-0.862	-0.8467	-0.8305	2.3892	3.7085	7.6077
	5	-0.8905	-0.8765	-0.8607	1.8005	2.6284	5.4698
	6	-0.9099	-0.8971	-0.8786	1.48	2.2923	4.6491
	7	-0.9223	-0.9082	-0.8926	1.2328	1.9345	3.333
1.5	3	-0.8669	-0.8456	-0.824	3.3486	5.3569	15.1194
	4	-0.9356	-0.9088	-0.8861	2.1219	3.322	6.8758
	5	-0.9802	-0.9511	-0.9266	1.9252	2.8344	5.6475
	6	-1.0571	-1.0233	-0.9952	1.4929	2.2312	4.4454
	7	-1.0372	-1.0067	-0.9804	1.4816	2.1252	4.3192
2	3	-0.9028	-0.871	-0.8416	3.4915	5.707	15.7507
	4	-0.9896	-0.9512	-0.9185	2.2748	3.4567	7.6392
	5	-1.055	-1.0167	-0.9836	2.0403	3.1359	6.038
	6	-1.0869	-1.045	-1.0122	1.7601	2.4463	4.8687
	7	-1.1222	-1.0739	-1.0371	1.496	2.1448	3.9157
2.5	3	-0.9316	-0.8871	-0.8563	3.6244	5.9548	14.2394
	4	-1.0217	-0.9752	-0.9431	2.413	3.6608	8.2219
	5	-1.094	-1.0416	-1.0061	1.9683	3.0045	5.654
	6	-1.1401	-1.0858	-1.0497	1.7171	2.5237	4.6132
	7	-1.1781	-1.1216	-1.0852	1.6862	2.3604	4.3077
3	3	-0.9487	-0.9064	-0.8723	3.8848	6.262	15.3208
	4	-1.0566	-1.0007	-0.9592	2.4537	3.635	8.088
	5	-1.1243	-1.0571	-1.0118	2.1611	3.1078	6.6269
	6	-1.1807	-1.12	-1.0732	1.8243	2.6557	5.2907
	7	-1.2323	-1.1633	-1.113	1.7411	2.3974	4.338
3.5	3	-0.9694	-0.9157	-0.8802	3.6677	5.5475	14.706
	4	-1.0751	-1.0194	-0.9791	2.4833	3.7404	8.2108
	5	-1.1445	-1.0823	-1.0383	2.2466	3.3349	6.8823
	6	-1.2051	-1.1371	-1.0861	1.9678	2.7908	5.4339
	7	-1.2535	-1.1855	-1.137	1.6821	2.456	4.6229
4	3	-0.9834	-0.9309	-0.892	3.7529	5.8408	16.5728
	4	-1.0987	-1.0433	-0.9977	2.5156	3.7436	7.5572
	5	-1.1785	-1.1083	-1.054	2.1927	3.141	6.1287
	6	-1.239	-1.1618	-1.1072	1.9349	2.8582	5.1429
	7	-1.2884	-1.2083	-1.153	1.8191	2.5726	4.8888

Table 2.9: Upper Record values, BLUE's of μ and σ for $\alpha = 1.5$

n	Upper record values	μ^*	σ^*
3	1.2940,2.5177,3.4619	0.0679	1.0106
4	0.9959,1.2717,2.1923,3.4995	0.0133	0.789
5	1.1460,1.2020,2.9643,3.4555,4.9878	-0.0026	0.9176
6	1.1584,1.7404,2.2788,2.3974,2.6612,5.8204	-0.0019	0.8945
7	0.8786,2.8224,3.1445,3.4072,3.8070,3.8294,5.5368	0.001	0.9528

Table 2.10: 95% Confidence interval for μ and σ

n	3	4	5	6	7
95%CI for μ (σ known)	(-2.2035,3.0636)	(-1.5718,2.082)	(-1.4505,1.7336)	(-1.0756,1.5012)	(-0.9432,1.2191)
95%CI for μ (σ unknown)	(-7.5538,1.2710)	(-3.4445,0.9593)	(-3.3021,1.1046)	(-2.4765,1.1330)	(-2.4674,1.1703)
95%CI for σ	(0.3489,3.9019)	(0.398,2.4687)	(0.5747,2.6925)	(0.6747,2.7188)	(0.844,3.18)

2.6 Prediction for future records

Prediction of future records becomes a problem of great interest. For example, while studying the record rainfall or snowfall, having observed the record values until the present time, we will be naturally interested in predicting the amount of rainfall or snowfall to be expected when the present record is broken for the first time in future. The best linear unbiased predicted value of the next record can be written as (see Balakrishnan and Chan, 1998)

$$y_{u(n)} = \mu^* + \sigma^* \beta_n$$

where μ^* and σ^* are the BLUE's based on the first (n-1) records and β_n is the n^{th} moment of record values. Prediction of next upper record value is obtained from a simulated data and presented in Table 2.11.

Table 2.11: Predicted records

n	simulated records of size (n-1)	Predicted value
3	3.5612,4.6331	5.6538
4	0.2391,2.1381,3.23	4.664
5	0.3232,1.1896,1.6908,3.068	3.93
6	0.1088,0.5235,1.2031,1.2407,3.239	3.942
7	1.1068,1.6784,1.9831,3.3077,3.4781,3.5190	3.5387
8	1.6361,2.4535,3.1141,3.2858,3.6631,3.7992,4.2374	4.2723

2.7 Entropy of Record value distribution

Entropy provides an excellent tool to quantify the amount of information (or uncertainty) contained in a random observation regarding its parent distribution. Shannon's(1948) entropy of an absolutely continuous random variable X with probability density function f(x) is given by

$$H_x[f(x)] = - \int_{-\infty}^{\infty} f(x) \ln[f(x)] dx$$

The entropy is always non-negative in the case of a discrete random variable X and is also invariant under a one-to- one transformation of X. For a continuous random variable, entropy is not invariant under a one-to-one transformation of X and it takes values in $(-\infty, +\infty)$.The entropy for some commonly used probability distributions have been tabulated by many authors. More recently Ebrahmi(2000) have explored the properties of entropy,Kullback - Leibler information and mutual information for order statistics. Now we discuss the entropy for the record values of MOEE (α, σ) . Let $H(R_n)$ be the entropy of the n^{th} record value. Then by Shakil (2005)

$$H(R_n) = \ln(\Gamma n) - (n - 1)\psi(n) - \frac{1}{\Gamma(n)} \int_{-\infty}^{\infty} [-\ln(1 - G(x))]^{n-1} g(x) \ln(g(x)) dx \quad (2.7.1)$$

where $\int_0^{\infty} t^{j-1} e^{-t} dt = \Gamma(j)$ and $\int_0^{\infty} t^{j-1} e^{-t} \ln(t) dt = \Gamma(j)\psi(j)$ $\psi(j)$ is the digamma function. For $n = 1$ entropy of the first record value is same as the entropy of parent distribution .Comparison of the entropy of parent distribution and n^{th} record value $n \geq 2$ is same as comparison of entropy of first record value with entropy of a given n^{th}

Table 2.12: Entropy of MOEE (α, σ)

Record	$\sigma = 0.5$	$\sigma = 1$	$\sigma = 2$	$\sigma = 5$
2	2.2113	1.5182	0.8250	-0.0913
4	2.0090	2.7021	1.3159	0.3996
6	2.2545	2.9476	1.5613	0.6450
8	2.4167	3.1098	1.7235	0.8073
10	2.5384	3.2315	1.8452	0.9289

record value .Since the first observation from the parent distribution is always considered as a record value, entropy of the first non-trivial record value is obtained when $n \geq 2$.

Theorem 2.7.1. For MOEE (α, σ) distribution if $H_{(j)}$ represents the entropy corresponding to j^{th} record, then

$$H_{(j)} = \ln(\Gamma j) - (j - 1)\psi(j) + j - \ln(\sigma) + \sum_{i=1}^{\infty} \frac{k^i}{i(i + 1)^j} \quad (2.7.2)$$

Proof By (2.7.1) the entropy of j^{th} record for MOEE (α, σ) is

$$H_{(j)} = \ln(\Gamma j) - (j - 1)\psi(j) - \frac{1}{\Gamma(j)} \int_0^{\infty} \left[-\ln \left(\frac{\alpha}{e^{\sigma x} - \alpha} \right) \right]^{j-1} v(x) \ln v(x) dx$$

where $v(x) = \frac{\alpha \sigma e^{\sigma x}}{(e^{\sigma x} - \alpha)^2}$. By the transformation $t = -\ln \frac{\alpha}{e^{\sigma x} - \alpha}$ and writing

$\ln(1 - ke^{-t}) = -\sum_{i=1}^{\infty} \frac{k^i e^{-it}}{i}$ where $k = 1 - \frac{1}{\alpha}$ the result(2.3.8) can be easily obtained.

Using (2.7.2) the entropy of MOEE (α, σ) for $\alpha = 0.8$ and for various record values and various values of σ are tabulated and presented in table (2.12).

It is evident that the sequence $\{H_j\}$ of entropies for record values is monotone increasing in j .

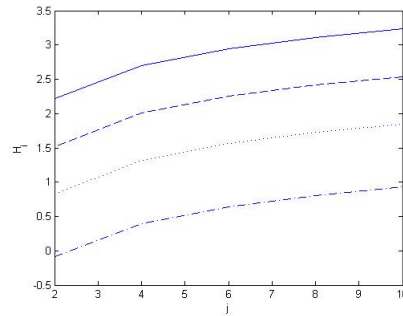


Figure 2.2: Graph of entropy of the j th record value, when $\alpha = 0.8$ and for various values of $\sigma = 0.5$ (solid), 1 (slash), 2 (dot) and 5 (slash dot) respectively.

2.8 Stress-Strength Analysis and Estimation of Reliability

Stress-strength analysis is an area in reliability theory where we assess the impact of stress on strength of devices and systems. It is measured by the expression $R = P(Y > X)$ which gives the reliability of a component in terms of the probability that the random variable X representing stress experienced by the component exceeds Y , representing the strength of the component. If stress exceeds strength, the component would fail, and vice versa. Kotz et al (2003) gives a detailed description of stress-strength theory. For more details one can see Nadarajah (2004), Kundu and Raqab (2009), and Bindu (2011) etc. Gupta et al (2010) showed that for two independent random variables represent strength (X) and stress (Y) follow the same Marshall-Olkin extended distributions with tilt parameters α_1 and α_2 then the Reliability of the system given by $P(X > Y)$ denoted by R is

$$R = \frac{\frac{\alpha_1}{\alpha_2}}{\left(\frac{\alpha_1}{\alpha_2} - 1\right)^2} \left[-\ln \frac{\alpha_1}{\alpha_2} + \frac{\alpha_1}{\alpha_2} - 1 \right]$$

To estimate R it is enough if we estimate α_1, α_2 by the method of m.l.e. The log likelihood equation here is

$$LL \propto m \ln(\alpha_1) + n \log(\alpha_2) - 2 \sum_{i=1}^m \log(e^{\sigma x_i} - (1 - \alpha_1)) - 2 \sum_{i=1}^n \log(e^{\sigma y_i} - (1 - \alpha_2))$$

Then the mle of α_1 and α_2 are the solutions of the non-linear equations

$$\begin{aligned}\frac{\partial LL}{\partial \alpha_1} &= \frac{m}{\alpha_1} - 2 \sum_{i=1}^m \frac{1}{(e^{\sigma x_i} - (1 - \alpha_1))} \\ \frac{\partial LL}{\partial \alpha_2} &= \frac{n}{\alpha_2} - 2 \sum_{i=1}^m \frac{1}{(e^{\sigma y_i} - (1 - \alpha_2))}\end{aligned}$$

By the property of m.l.e for $m \rightarrow \infty, n \rightarrow \infty$

$$\sqrt{m}(\hat{\alpha}_1 - \alpha_1), \sqrt{n}(\hat{\alpha}_2 - \alpha_2) \xrightarrow{d} N_2(\mathbf{0}, \text{diag}\{\frac{1}{a_{11}}, \frac{1}{a_{22}}\})$$

$$\text{where } a_{11} = \lim_{m,n \rightarrow \infty} \frac{1}{m} I_{11} = \frac{1}{3\alpha_1^2} \text{ and } a_{22} = \lim_{m,n \rightarrow \infty} \frac{1}{n} I_{22} = \frac{1}{3\alpha_2^2}$$

Now the information matrix has the elements

$$\begin{aligned}I_{11} &= -E \left(\frac{\partial^2 LL}{\partial \alpha_1^2} \right) \\ &= -E \left(\frac{-m}{\alpha_1^2} + 2 \sum_{i=1}^m \frac{1}{(e^{\sigma x_i} - (1 - \alpha_1))^2} \right) \\ &= \frac{m}{\alpha_1^2} - 2\alpha_1 m \int_{\alpha}^{\infty} \frac{dt}{t^4} \\ &= \frac{m}{3\alpha_1^2}\end{aligned}$$

$$\text{similarly } I_{22} = -E \left(\frac{\partial^2 LL}{\partial \alpha_2^2} \right) = -\frac{n}{3\alpha_2^2} \text{ and } I_{12} = I_{21} = -E \left(\frac{\partial^2 LL}{\partial \alpha_1 \partial \alpha_2} \right) = 0$$

Now from Gupta et al(2009)the 95% confidence interval for R is given by

$$\hat{R} \mp 1.96 \hat{\alpha}_1 b_1(\hat{\alpha}_1, \hat{\alpha}_2) \sqrt{\frac{3}{m} + \frac{3}{n}},$$

where $b_1(\alpha_1, \alpha_2) = \frac{\partial R}{\partial \alpha_1} = \frac{\alpha_2}{(\alpha_1 - \alpha_2)^3} \left[-2(\alpha_1 - \alpha_2) + (\alpha_1 + \alpha_2) \ln \frac{\alpha_1}{\alpha_2} \right]$ and

$$b_2(\alpha_1, \alpha_2) = \frac{\partial R}{\partial \alpha_2} = \frac{\alpha_1}{(\alpha_1 - \alpha_2)^3} \left[2(\alpha_1 - \alpha_2) - (\alpha_1 + \alpha_2) \ln \frac{\alpha_1}{\alpha_2} \right] = -\frac{\alpha_1}{\alpha_2} b_1(\alpha_1, \alpha_2)$$

2.8.1 Simulation Study

We generate $N=10,000$ sets of X -samples and Y -samples from the Marshall-Olkin extended exponential distribution with parameters α_1, σ and α_2, σ respectively. The combinations of samples of sizes $m = 20, 25, 30$ and $n = 20, 25, 30$ along with $m = 40, n = 40$ are considered. The validity of the estimate of R is discussed by the measures namely average bias of the estimate (\bar{b}), average mean square error of the estimate (AMSE), average confidence interval of the estimate and coverage probability.

The numerical values obtained for the measures listed above are presented in tables 2.13-2.16. For $\alpha_1 < \alpha_2$ the average bias is positive and for $\alpha_1 > \alpha_2$ the average bias is negative but in both cases the average bias decreases as the sample size increases. The average MSE is almost symmetric with respect to (α_1, α_2) . This symmetric property can also be observed in the case of average confidence interval and its performance is quite good. The coverage probability is very close to 0.95 and approaches to the nominal value as the sample size increases. The simulation study indicates that the average bias, average MSE, average confidence interval and coverage probability do not show much variability for various parameter combinations.

2.9 Applications in Autoregressive Time Series Modelling

One of the simplest and widely used time series models is the autoregressive models and it is well known that autoregressive process of appropriate orders is extensively used for modeling time series data. The p^{th} order autoregressive model is defined by

$$X_n = a_1 X_{n-1} + a_2 X_{n-2} + \dots + a_p X_{n-p} + \epsilon_n$$

where $\{\epsilon_n\}$ is a sequence of independent and identically distributed random variables and a_1, a_2, \dots, a_n are autoregressive parameters. In particular the first order autoregressive model is

$$X_n = a_1 X_{n-1} + \epsilon_n, n = 1, 2, \dots, |a_1| < 1$$

Table 2.13: Average bias and average MSE of the simulated estimates of R for $\sigma = 0.5$

(α_1, α_2)								
(m,n)	Average bias (\bar{b})				Average Mean Square Error AMSE			
	(0.5,0.8)	(0.8,1.5)	(0.8,0.5)	(1.5,0.8)	(0.5,0.8)	(0.8,1.5)	(0.8,0.5)	(1.5,0.8)
(20,20)	0.0433	0.0586	-0.0432	-0.0582	0.0063	0.0065	0.0063	0.0064
(20,25)	0.0432	0.0578	-0.0455	-0.0582	0.0061	0.0061	0.0062	0.0061
(20,30)	0.0424	0.0574	-0.0481	-0.0599	0.0057	0.0059	0.0062	0.0060
(25,20)	0.0455	0.0593	-0.0423	-0.0579	0.0062	0.0063	0.0060	0.0061
(25,25)	0.0451	0.0584	-0.0468	-0.0585	0.0058	0.0058	0.0059	0.0060
(25,30)	0.0438	0.0576	-0.0478	-0.0593	0.0055	0.0057	0.0056	0.0058
(30,20)	0.0475	0.0596	-0.0430	-0.0573	0.0061	0.0060	0.0057	0.0059
(30,25)	0.0473	0.0587	-0.0450	-0.0585	0.0056	0.0058	0.0056	0.0058
(30,30)	0.0463	0.0580	-0.0465	-0.0596	0.0054	0.0056	0.0053	0.0056
(40,40)	0.0458	0.0575	-0.0468	-0.0597	0.0048	0.0052	0.0048	0.0052

Table 2.14: Average confidence length and coverage probability of the simulated 95% confidence intervals of R for $\sigma = 0.5$

(α_1, α_2)								
(m,n)	Average confidence length				coverage probability			
	(0.5,0.8)	(0.8,1.5)	(0.8,0.5)	(1.5,0.8)	(0.5,0.8)	(0.8,1.5)	(0.8,0.5)	(1.5,0.8)
(20,20)	0.3506	0.3516	0.3508	0.3516	0.9748	0.9763	0.9793	0.9740
(20,30)	0.3327	0.3336	0.3432	0.3432	0.9780	0.9719	0.9696	0.9685
(20,25)	0.3207	0.3209	0.3216	0.3223	0.9805	0.9716	0.9672	0.9624
(25,20)	0.3335	0.3344	0.3329	0.3337	0.9706	0.9693	0.9788	0.9726
(25,25)	0.3146	0.3152	0.3145	0.3154	0.9739	0.9620	0.9742	0.9618
(25,30)	0.3010	0.3019	0.3017	0.3023	0.9787	0.9572	0.9699	0.9545
(30,20)	0.3199	0.3222	0.3204	0.3211	0.9700	0.9600	0.9782	0.9718
(30,25)	0.3017	0.3023	0.3015	0.3019	0.9655	0.9537	0.9738	0.9582
(30,30)	0.2880	0.2883	0.2879	0.2883	0.9738	0.9469	0.9763	0.9468
(40,40)	0.2500	0.2502	0.2499	0.2502	0.9664	0.9108	0.9676	0.9164

Table 2.15: Average bias and average MSE of the simulated estimates of R for $\sigma = 3$

		(α_1, α_2)							
		Average bias (\bar{b})				Average Mean Square Error AMSE			
(m,n)		(0.5,0.8)	(0.8,1.5)	(0.8,0.5)	(1.5,0.8)	(0.5,0.8)	(0.8,1.5)	(0.8,0.5)	(1.5,0.8)
(20,20)		0.0433	0.0586	-0.0432	-0.0582	0.0063	0.0065	0.0063	0.0064
(20,25)		0.0432	0.0578	-0.0455	-0.0582	0.0061	0.0061	0.0062	0.0061
(20,30)		0.0424	0.0574	-0.0481	-0.0599	0.0057	0.0059	0.0062	0.0060
(25,20)		0.0455	0.0593	-0.0423	-0.0579	0.0062	0.0063	0.0060	0.0061
(25,25)		0.0451	0.0584	-0.0468	-0.0585	0.0058	0.0058	0.0059	0.0060
(25,30)		0.0438	0.0576	-0.0478	-0.0593	0.0055	0.0057	0.0056	0.0058
(30,20)		0.0475	0.0596	-0.0430	-0.0573	0.0061	0.0060	0.0057	0.0059
(30,25)		0.0473	0.0587	-0.0450	-0.0585	0.0056	0.0058	0.0056	0.0058
(30,30)		0.0463	0.0580	-0.0465	-0.0596	0.0054	0.0056	0.0053	0.0056
(40,40)		0.0458	0.0575	-0.0468	-0.0597	0.0048	0.0052	0.0048	0.0052

Table 2.16: Average confidence length and coverage probability of the simulated 95% confidence intervals of R for $\sigma = 3$

		(α_1, α_2)							
		Average confidence length				coverage probability			
(m,n)		(0.5,0.8)	(0.8,1.5)	(0.8,0.5)	(1.5,0.8)	(0.5,0.8)	(0.8,1.5)	(0.8,0.5)	(1.5,0.8)
(20,20)		0.3506	0.3512	0.3505	0.3512	0.9699	0.9799	0.9705	0.9819
(20,25)		0.3328	0.3334	0.3331	0.3339	0.9660	0.9785	0.9612	0.9794
(20,30)		0.3206	0.3210	0.3210	0.3214	0.9641	0.9764	0.9606	0.9739
(25,20)		0.3333	0.3339	0.3329	0.3334	0.9643	0.9762	0.9674	0.9799
(25,25)		0.3145	0.3150	0.3146	0.3150	0.9625	0.9728	0.9583	0.9733
(25,30)		0.3012	0.3016	0.3015	0.3017	0.9607	0.9713	0.9581	0.9677
(30,20)		0.3212	0.3215	0.3205	0.3211	0.9629	0.9753	0.9676	0.9762
(30,25)		0.2991	0.3018	0.3013	0.3017	0.9604	0.9692	0.9571	0.9704
(30,30)		0.2877	0.2880	0.2877	0.2880	0.9503	0.9660	0.9535	0.9655
(40,40)		0.2499	0.2498	0.2498	0.2498	0.9356	0.9485	0.9383	0.9479

The need for non-Gaussian autoregressive models have been long felt from the fact that many naturally arising time series are clearly non-Gaussian with Markovian dependence structure. Many non - Gaussian autoregressive processes were introduced and studied during the past two decades (see Jayakumar et al. (1995), Jose and Pillai (1995)). Jayakumar and Pillai (1993) introduced and studied first order autoregressive Mittag-Leffler process. Pillai and Jayakumar (1995) characterized a p^{th} order autoregressive Mittag-Leffler process using specialized class L property. Jose and Pillai (1995) developed generalized autoregressive time series models in Mittag-Leffler variables. Jose and Seethalakshmi (2004) studied geometric Mittag-Leffler processes. Jose and Alice (2001), Alice and Jose (2003, 2004 a,b,c, 2005 a,b) developed autoregressive minification processes and studied their properties.

Now we discuss some application of MOEE distribution in autoregressive time series modeling.

Lewis and McKenzie (1991) introduced and discussed minification processes having structure

$$X_n = \min(aX_{n-1}, \epsilon_n), \quad n = 1, 2, \dots, |a_1| < 1$$

2.9.1 An AR (1) model with MOEE marginal distribution

We construct a first order autoregressive minification process with a more general structure given by (2.9.1). The model is developed as follows. Consider an AR (1) structure

$$X_n = \begin{cases} \epsilon_n & \text{with probability } p \\ \min(X_{n-1}, \epsilon_n) & \text{with probability } (1-p); \quad 0 \leq p \leq 1 \end{cases} \quad (2.9.1)$$

where $\{\epsilon_n\}$ is a sequence of i.i.d. r.v.s with exponential distribution with unit mean and is independent of $\{X_n\}$.

Theorem 5.1 Consider the AR (1) structure given by (2.9.1). Then $\{X_n\}$ is stationary Markovian with MOEE marginal distribution if $\{\epsilon_n\}$ is distributed as exponential distribution

with unit mean.

Proof From (2.9.1) it follows that

$$\bar{F}_{X_n}(x) = p\bar{F}_{\varepsilon_n}(x) + (1-p)\bar{F}_{X_{n-1}}(x)\bar{F}_{\varepsilon_n}(x) \quad (2.9.2)$$

Under stationary equilibrium

$$\bar{F}_X(x) = \frac{p\bar{F}_{\varepsilon}(x)}{1 - (1-p)\bar{F}_{\varepsilon}(x)} \quad \text{and hence} \quad \bar{F}_{\varepsilon}(x) = \frac{\bar{F}_X(x)}{p + (1-p)\bar{F}_X(x)}.$$

If $\varepsilon_n \sim \text{Exp}(1)$, $\bar{F}_{\varepsilon}(x) = e^{-x}$, then it easily follows that,

$$\bar{F}_X(x) = \frac{pe^{-x}}{1 - (1-p)e^{-x}},$$

which is the survival function of MOEE (p).

Conversely, if we take,

$$\bar{F}_{X_n}(x) = \frac{pe^{-x}}{1 - (1-p)e^{-x}},$$

it is easy to show that $\bar{F}_{\varepsilon_n}(x)$ is distributed as $\text{Exp}(1)$ and the process is stationary.

In order to establish stationarity, we proceed as follows. Assume $X_{n-1} \stackrel{d}{=} \text{MOEE}(p)$ and $\varepsilon_n \stackrel{d}{=} \text{Exp}(1)$, then from (11),

$$\bar{F}_{X_n}(x) = \frac{pe^{-x}}{1 - (1-p)e^{-x}}.$$

This establishes that $\{X_n\}$ is distributed as MOEE (p).

Even if X_0 is arbitrary, it is easy to establish that $\{X_n\}$ is stationary and is asymptotically marginally distributed as MOEE (p).

In order to study the behavior of the process we simulate the sample paths for various

values of p . From the sample path properties it follows that the MOEE AR(1) minification process can be used for modeling a rich variety of real data from various contexts such as financial modeling, reliability modeling, hydrological modeling etc.

Now we consider some properties of MOEE AR(1) minification processes we start with

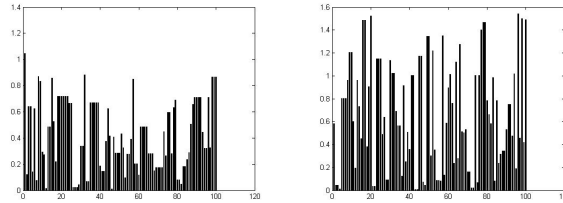


Figure 2.3: Sample paths of MOEE AR (1) process $p=0.6,0.8$

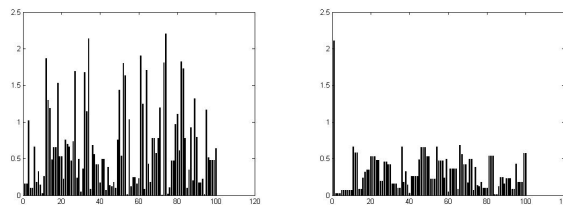


Figure 2.4: Sample paths of MOEE AR (1) process $p= 0.9, 0.5$

the joint survival function of the random variables X_{n+1} and X_n .

Let $\bar{S}(x, y) = P(X_{n+1} > x, X_n > y)$ be the joint survival function of the random variables X_{n+1} and X_n . Then we have

$$\begin{aligned} \bar{S}(x, y) &= p\bar{F}_\varepsilon(x)\bar{F}_X(y) + (1-p)\bar{F}_\varepsilon(x)\bar{F}_X(\max(x, y)) \\ &= \begin{cases} \bar{F}_\varepsilon(x)\bar{F}_X(y) & y > x \\ \bar{F}_\varepsilon(x)(p\bar{F}_X(y) + (1-p)\bar{F}_X(x)), & y < x \end{cases} \end{aligned}$$

$$= \begin{cases} \frac{pe^{-x-y}}{1-(1-p)e^{-y}}, & y > x \\ \frac{pe^{-x}(pe^{-y}+(1-p)e^{-x}-2p(1-p)e^{-x-y})}{(1-(1-p)e^{-x})(1-(1-p)e^{-y})}, & y < x \end{cases}$$

The joint survival function \bar{S} is not absolutely continuous since the probability $P(X_{n+1} = X_n)$ is positive. Namely, it is easy to show that

$$P(X_{n+1} = X_n) = \frac{-p(1-p+\log p)}{(1-p)^2} \in (0, 0.5)$$

Consider now the probability of the event $\{X_{n+1} > X_n\}$. From (2.9.1) it follows that

$$P(X_{n+1} > X_n) = pP(\varepsilon_{n+1} > x_n) = \frac{p(1-p+p\log p)}{(1-p)^2} \in (0, 0.5)$$

Also, we can show that

$$P(X_{n+2} > X_n) = \frac{p(2-p-p^2+3p\log p)}{(1-p)^2} \in (0, 0.5)$$

We can use these probabilities to estimate the unknown parameter p . Define the random variables $U_n = I(X_{n+1} > X_n)$ and $V_n = I(X_{n+2} > X_n)$. It is easy to show that $E(U_n) = P(X_{n+1} > X_n)$ and $E(V_n) = P(X_{n+2} > X_n)$. Now we consider the equations

$$\begin{aligned} \frac{1}{N} \sum_{i=1}^N U_i &= \frac{p(1-p+p\log p)}{(1-p)^2} \\ \frac{1}{N-1} \sum_{i=1}^{N-1} V_i &= \frac{p(2-p-p^2+3p\log p)}{(1-p)^2} \end{aligned}$$

Solving these equations, we will obtain that the estimator of the unknown parameter p is given by

$$\hat{p} = \frac{3}{N} \sum_{i=1}^N U_i - \frac{1}{N-1} \sum_{i=1}^{N-1} V_i$$

Since the MOEE AR(1) minification process $\{X_n\}$ is ergodic, it follows that \hat{p} is consistent

estimator for p .

In table 2.6 we give some numerical results of the estimation. We estimate 10000 realizations of the MOEE AR (1) minification process for the true values $p = 0.2, p = 0.4, p = 0.6$ and $p = 0.8$. The simulations are repeated 100 times. We computed the sample means and the standard errors of the estimate of \hat{p} .

Let us consider the autocovariance function at lag 1. After some calculations we obtain that

$$E(X_{n+1}X_n) = p \int_0^{\infty} \frac{xe^{-x} dx}{1 - (1-p)e^{-x}} = \frac{p}{1-p} \cdot Li_2(1-p),$$

where

$$Li_2(z) = z \int_0^{\infty} \frac{xe^{-x} dx}{1 - ze^{-x}}$$

is dilogarithm. Now, autocovariance function at lag 1 is

$$cov(X_{n+1}, X_n) = \frac{p}{1-p} \cdot Li_2(1-p) - \frac{p^2 \log p}{(1-p)^2}$$

the autocorrelation function at lag 1 is

$$Corr(X_{n+1}, X_n) = \frac{p(1-p)Li_2(1-p) - p^2 \log p}{2p(1-p)Li_2(1-p) - p^2 \log p}$$

The autocovariance function and the autocorrelation function at lag 1 for various values of p are given at figure ??.

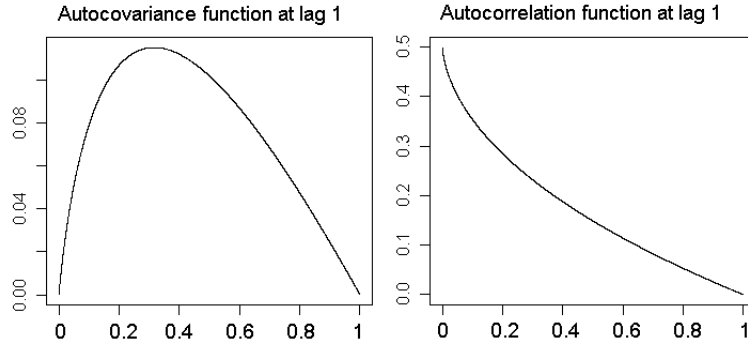


Figure 2.5: Autocovariance and autocorrelation function at lag 1 for various values of p

2.10 Extension to k^{th} order processes

In this section we develop a k^{th} order autoregressive model. Consider an autoregressive model of order k with structure as

$$X_n = \begin{cases} \varepsilon_n & \text{w. p. } p_0 \\ \min(X_{n-1}, \varepsilon_n) & \text{w.p. } p_1 \\ \vdots & \\ \min(X_{n-k}, \varepsilon_n) & \text{w.p. } p_k. \end{cases}$$

such that $0 < p_i < 1$, $p_1 + p_2 + \dots + p_k = 1 - p_0$; where $\{\varepsilon_n\}$ is a sequence of i.i.d r.v.s following MOEE distribution independent of $\{X_{n-1}, X_{n-2}, \dots\}$.

$$\overline{F}_{X_n}(x) = p_0 \overline{F}_{\varepsilon_n}(x) + p_1 \overline{F}_{X_{n-1}}(x) \overline{F}_{\varepsilon_n}(x) + \dots + p_k \overline{F}_{X_{n-k}}(x) \overline{F}_{\varepsilon_n}(x)$$

Under stationary equilibrium,

$$\overline{F}_X(x) = p_0 \overline{F}_\varepsilon(x) + p_1 \overline{F}_X(x) \overline{F}_\varepsilon(x) + \dots + p_k \overline{F}_X(x) \overline{F}_\varepsilon(x)$$

Table 2.17: Some numerical results of the estimation

n	$\hat{p}(\text{True } p = 0.2)$	$SE(\hat{p})$	$\hat{p}(\text{True } p = 0.4)$	$SE(\hat{p})$
100	0.205328	0.050357	0.391892	0.067521
500	0.202402	0.023738	0.397680	0.029513
1000	0.200810	0.017016	0.396480	0.022201
5000	0.200076	0.006890	0.398344	0.011454
10000	0.200169	0.004981	0.399264	0.007092
n	$\hat{p}(\text{True } p = 0.6)$	$SE(\hat{p})$	$\hat{p}(\text{True } p = 0.8)$	$SE\hat{p}$
100	0.596091	0.084100	0.796522	0.097800
500	0.595670	0.034729	0.801667	0.038394
1000	0.595555	0.025793	0.802864	0.030903
5000	0.599764	0.010598	0.801462	0.015817
10000	0.598863	0.007853	0.800391	0.011084

This reduces to

$$\bar{F}_X(x) = \frac{p_0 \bar{F}_\epsilon(x)}{1 - (1 - p_0) \bar{F}_\epsilon(x)}$$

This shows that theorem 5.1 can be suitably extended to this case also.

2.11 Application

Let x_1, x_2, \dots, x_n be a random sample of size n from $MOEE(\alpha, \sigma)$ distribution with p.d.f (2.2.2). The likelihood function is given by

$$L(\alpha, \sigma) = \frac{(\alpha\sigma)^n e^{n\sigma\bar{x}}}{\prod_{i=1}^n (e^{\sigma x_i} - (1 - \alpha))^2}$$

Table 2.18: Summary of fitting for the MOEE and exponential distribution to the failure times of the air conditioning system of an airplane.

Model	Parameter	MLE	-Log-likelihood	K-S statistic	p-value
Exponential	σ	0.0168	152.6297	0.2128	0.132
MOEE	α	0.4072	151.425	0.1293	0.6978
	σ	0.0106			

The m.l.e's of α and σ are given by the solution of the two equations.

$$\frac{n}{\alpha} - 2 \sum_{i=1}^n \frac{1}{e^{\sigma x_i} - \alpha} = 0 \quad (2.11.1)$$

and

$$\frac{n}{\sigma} + n\bar{x} - 2 \sum_{i=1}^n \frac{x_i e^{\sigma x_i}}{e^{\sigma x_i} - \alpha} = 0 \quad (2.11.2)$$

When $\alpha = 1$, ie, for exponential distribution $\hat{\sigma} = \frac{1}{\bar{x}}$

Here we show that the extended model of exponential distribution can be a better model than the one parameter exponential model when it is fitted for the following data .

Data set : (Linhart and Zucchini (1986, page 69)). The following data are failure times of the air conditioning system of an airplane: 23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95.

Using R program we estimate the parameters and draw the PP plot and QQ plot for the data. It is clear that the extended exponential distribution significantly improves the fit given by exponential distribution with single parameter.

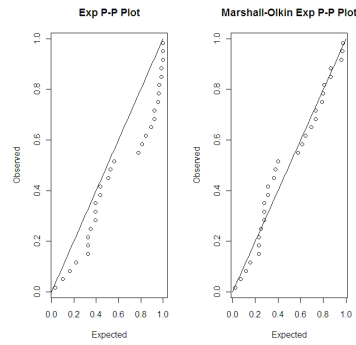


Figure 2.6: PP plot of Exponential model versus MOEE

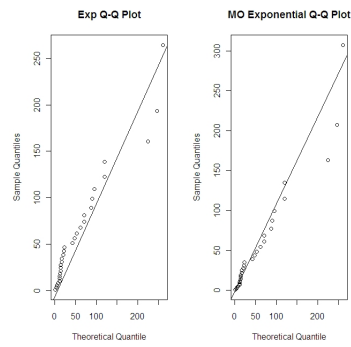


Figure 2.7: QQ plot of Exponential model versus MOEE

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CHAPTER 3

Marshall-Olkin Extended Fréchet Distribution and its properties

3.1 Introduction

The Extreme value distribution of type II was named after Fréchet(1927), who devised one possible limiting distribution for a sequence of maxima, provided convenient scale normalization. Extreme Value distribution, are widely used in risk management, Finance, insurance, economics, hydrology, material sciences, telecommunications and many other industries dealing with extreme events. In recent years the Generalised Extreme value theory has become increasingly familiar in economics, and particularly in financial econometrics, because of its role in quantifying the probabilities of extreme falls in the value of financial funds. Harter(1978) prepared an authoritative bibliography of extreme value theory which is of substantial scientific work.Beirlant et al(1996) provide a lucid practical analysis of extrme values with emphasis on actuarial applications. More information about the Fréchet distribution can be found in Resnick(1987),Kotz and Nadarajah

(2000), Coles(2001), Johnson et al (2004), De Haan and Ferreira(2006) and Resis and Thomas(2007). Recently, some generalizations of the Fréchet distribution are considered. Nadarajah and Kotz (2003) introduced the exponentiated Fréchet distribution with distribution function $F(x) = 1 - \left(1 - e^{-(\delta/x)^\beta}\right)^\alpha$, $x, \delta, \beta, \alpha > 0$. For $\alpha = 1$ the exponentiated Fréchet distribution becomes the Fréchet distribution with parameters δ and β . Abd-Elfattah et al (2010) discussed goodness of fit tests for Generalized Fréchet Distribution. Nadarajah and Gupta (2004) introduced the beta Fréchet distribution with the distribution function

$$F(x) = \frac{1}{B(a, b)} \int_0^{e^{-(\delta/x)^\beta}} w^{a-1} (1-w)^{b-1} dw, \quad x, \delta, \beta, a, b > 0.$$

The beta Fréchet distribution generalizes some well know distributions. For $a = 1$, we obtain the exponentiated Fréchet distribution with parameters δ, β and $\alpha = b$. For $a = 1$ and $b = 1$ we obtain the Fréchet distribution with parameters δ and β .

In this chapter we introduce a new distribution called Marshall–Olkin Extended Fréchet distribution is introduced Some properties of the newly introduced distribution such as the shapes of the probability density and hazard rate functions etc. are considered. The expression for the moments are given, the Rényi entropy is derived, the density of the order statistics are derived and the parameters are estimated by the method of maximum likelihood. We analyze a real data set and compare our distribution with the Fréchet, exponentiated Fréchet and beta Fréchet distributions.

3.2 Marshall-Olkin Extended Fréchet distribution

Now we consider the Fréchet distribution with survival function $\bar{F}(x) = 1 - e^{-(\frac{\delta}{x})^\beta}$; $x, \delta, \beta > 0$. Then applying (1.2.1), we get a new family of distribution called the Marshall-

Olkin Extended Fréchet distribution distribution with survival function given by

$$\overline{G}(x) = \frac{\alpha[1 - e^{-(\delta/x)^\beta}]}{\alpha + (1 - \alpha)e^{-(\delta/x)^\beta}}, \quad x, \delta, \beta, \alpha > 0. \quad (3.2.1)$$

We shall refer to as MOEFR family and we will denote with $\text{MOEFR}(\delta, \beta, \alpha)$ the fact that a random variable has a MOEFR distribution with parameters δ , β and α . Let us consider some properties of the distributions from the MOEFR family.

3.2.1 The probability density function

The probability density function is

$$g(x; \delta, \beta, \alpha) = \frac{\alpha\beta \left(\frac{\delta}{x}\right)^{\beta+1} e^{-(\delta/x)^\beta}}{\delta \left(\alpha + (1 - \alpha)e^{-(\delta/x)^\beta}\right)^2}, \quad x, \delta, \beta, \alpha > 0.$$

The probability density function is unimodal and this result follows from the following theorem.

Theorem 3.2.1. *The probability density function of the MOEFR distribution has a unique mode at $x = x_0$, where x_0 is the solution of the equation*

$$\alpha(\beta + 1) - \alpha\beta t(x) + (\beta + 1)(1 - \alpha)e^{-t(x)} + \beta(1 - \alpha)t(x)e^{-t(x)} = 0,$$

with $t(x) = \delta^\beta x^{-\beta}$.

Proof: The first derivative of the function $\log g(x)$ can be written as

$$(\log g(x))' = \frac{t'(x)s(x)}{\beta t(x)(\alpha + (1 - \alpha)e^{-t(x)})},$$

where $s(x) = \alpha(\beta + 1) - \alpha\beta t(x) + (\beta + 1)(1 - \alpha)e^{-t(x)} + \beta(1 - \alpha)t(x)e^{-t(x)}$. The first derivative of the function $s(x)$ can be written as $s'(x) = t'(x)u(x)$, where $u(x) = -\alpha\beta - (1 - \alpha)e^{-t(x)} - \beta(1 - \alpha)t(x)e^{-t(x)}$. We can see that for $0 < \alpha < 1$, the function $u(x)$ is negative and since $t'(x)$ is negative, it follows that $s'(x) > 0$. This implies that the

function $s(x)$ increases and since $s(0) = -\infty$ and $s(\infty) = \beta + 1$, it follows that function $s(x)$ has a unique root, say at $x = x_0$. From this it follows that the function $g'(x)$ is positive for $x < x_0$ and negative for $x > x_0$. Thus in the case $0 < \alpha < 1$, we obtain that the pdf is unimodal with mode at $x = x_0$. Now we consider the case $\alpha > 1$. If $0 < \beta < (\alpha - 1)/\alpha$, then $u(x)$ is an increasing function with $u(0) = -\alpha\beta$ and $u(\infty) = -1 + \alpha - \alpha\beta > 0$. Thus $u(x)$ has a unique root, say x_1 . Then the function $s(x)$ increases on $(0, x_0]$ and decreases on (x_0, ∞) . Since $s(0) = -\infty$ and $s(\infty) = \beta + 1$, it follows that $s(x)$ has a unique root, say at $x = x_0$. As in the case $0 < \alpha < 1$, we can conclude that $g(x)$ is unimodal function. If $(\alpha - 1)/\alpha < \beta < 1$, then the function $u(x)$ is negative, which implies that the function $s(x)$ increases. Since $s(0) = -\infty$ and $s(\infty) = \beta + 1$, we can conclude as in the case $0 < \alpha < 1$, that the function $g(x)$ is unimodal. Finally, let us consider the case $\beta > 1$. Then the function $u(x)$ increases on $(0, x_2]$ and decreases on (x_2, ∞) , where x_2 is the solution of the equation $t(x) = 1 - 1/\beta$. Since $u(0) = -\alpha\beta$, $u(\infty) = -1 + \alpha - \alpha\beta > 0$ and $u(x_2) = -\beta [\alpha + (1 - \alpha)e^{-t(x_2)}] < -\beta < 0$, it follows that the function $u(x)$ is negative. This implies that the function $s(x)$ is increasing and the conclusion that the function $g(x)$ is unimodal follows as above.

3.2.2 The hazard rate function

The hazard rate function is

$$h(x) = \frac{\beta \left(\frac{\delta}{x}\right)^{\beta+1} e^{-(\delta/x)^\beta}}{\delta \left(\alpha + (1 - \alpha)e^{-(\delta/x)^\beta}\right) \left(1 - e^{-(\delta/x)^\beta}\right)}, \quad x, \delta, \beta, \alpha > 0.$$

The hazard rate function is upside-down bathtub shaped and this follows from the following theorem.

Theorem 3.2.2. *The hazard rate function is upside-down bathtub shaped with maximum in x_0 , where x_0 is the solution of the equation*

$$(\beta + 1) [\alpha + (1 - 2\alpha)e^{-t(x)}] - (1 - \alpha)e^{-2t(x)} [\beta + 1 + \beta t(x)] - \alpha\beta t(x) = 0, \quad (3.2.2)$$

where $t(x) = \delta^\beta x^{-\beta}$.

Proof: Let $v(x)$ be a left side of the equation (3.2.2). The first derivative of the function $\log h(x)$ is

$$(\log h(x))' = \frac{t'(x)v(x)}{\beta t(x)(\alpha + (1 - \alpha)e^{-t(x)})(1 - e^{-t(x)})}.$$

To prove that the hazard rate function is upside-down bathtub shaped it is sufficient to show that the function $v(x)$ is negative for $x < x_0$ and positive for $x > x_0$. The first derivative of the function $v(x)$ can be written as $v'(x) = t'(x)w(x)$, where

$$w(x) = -\alpha\beta - (\beta + 1)(1 - 2\alpha)e^{-t(x)} + (1 - \alpha)(\beta + 2)e^{-2t(x)} + 2(1 - \alpha)\beta t(x)e^{-2t(x)}.$$

Also, the first derivative of the function $w(x)$ can be written as $w'(x) = t'(x)e^{-t(x)}z(x)$, where

$$z(x) = (\beta + 1)(1 - 2\alpha) - 4(1 - \alpha)e^{-t(x)}(1 + \beta t(x)).$$

We can see that if $1/2 < \alpha < 1$, then the function $z(x)$ is negative, which implies that the function $w(x)$ increases with $w(0) = -\alpha\beta$ and $w(\infty) = 1$. Thus the function $w(x)$ has unique root, say x_1 . From this it follows that the function $v(x)$ increases on $(0, x_1]$ and decreases on (x_1, ∞) . Since $v(0) = -\infty$ and $v(\infty) = 0$, it follows that the function $v(x)$ has unique root, say x_0 . Also, we obtain that $v(x)$ is negative for $x < x_0$ and positive for $x > x_0$. Let us consider the case $0 < \alpha < 1/2$. Suppose first that $0 < \beta < 1$. Then the function $z(x)$ is increasing function with $z(0) = (1 - 2\alpha)(1 + \beta) > 0$ and $z(\infty) = -3 + 2\alpha + \beta - 2\alpha\beta < 0$. Thus $z(x)$ has unique root, say x_2 , which implies that the function $w(x)$ decreases on $(0, x_2]$ and increases on (x_2, ∞) . Since $w(0) = -\alpha\beta$ and $w(\infty) = 1$, it follows that the function $w(x)$ has unique root, say x_1 , with $w(x) < 0$ for $x < x_1$ and $w(x) > 0$ for $x > x_1$. This implies that the function $v(x)$ has the same behavior as in the case $1/2 < \alpha < 1$. Let $0 < \alpha < 1/2$ and $\beta > 1$. Then the function increases on $(0, x_3]$ and decreases on (x_3, ∞) , where x_3 is the solution of the equation $t(x) = 1 - 1/\beta$. As before, we have that $z(0) = (1 - 2\alpha)(1 + \beta) > 0$, but $z(\infty) = -3 + 2\alpha + \beta - 2\alpha\beta$ can be positive or negative. If $-3 + 2\alpha + \beta - 2\alpha\beta < 0$, then the function $z(x)$ has unique

root, say x_2 , with $z(x) > 0$ for $x < x_2$ and $z(x) < 0$ for $x > x_2$. Then the function $w(x)$ has the same behavior as in the case $0 < \alpha < 1/2$ and $0 < \beta < 1$. Thus from this it follows that $v(x)$ is negative for $x < x_0$ and positive for $x > x_0$. If $-3 + 2\alpha + \beta - 2\alpha\beta > 0$, then the function $z(x)$ has two roots, say x_{21} and x_{22} . From this it follows that the function $w(x)$ decreases on $(0, x_{21}] \cup (x_{22}, \infty)$ and increases on $(x_{21}, x_{22}]$. Since $w(0) = -\alpha\beta$ and $w(\infty) = 1$, we obtain that the functions $w(x)$ and $v(x)$ have the same behaviors as in the above case. Finally, let us consider the case $\alpha > 1$. As before, let us first consider the case $0 < \beta < 1$. It follows that the function $z(x)$ increases with $z(0) = (1 - 2\alpha)(1 + \beta) < 0$ and $z(\infty) = -3 + 2\alpha + \beta - 2\alpha\beta$ which can be positive or negative. It is easy to show that when $z(\infty) < 0$, the functions $w(x)$ and $v(x)$ have the same behaviors as in the case $1/2 < \alpha < 1$. If $z(\infty) > 0$, then the function $z(x)$ has unique root, say x_2 . This implies that the function $w(x)$ increases on $(0, x_2]$ and decreases on (x_2, ∞) . Since $w(0) = -\alpha\beta$ and $w(\infty) = 1$, it follows that the function $w(x)$ has unique root, say x_1 , with $w(x) < 0$ for $x < x_1$ and $w(x) > 0$ for $x > x_1$. This implies that the function $v(x)$ has the same behavior as in the case $1/2 < \alpha < 1$. Similarly as in the above cases, we can show that in the case $\alpha > 1$ and $\beta > 1$, the function $v(x)$ has the same behavior as in the case $1/2 < \alpha < 1$. Figure 3.1 gives the graphs of the functions $g(x)$ and $h(x)$ for different values of the parameters δ , β and α .

3.2.3 Moments, Quantiles and Rényi entropy

If a random variable X has MOEFR(δ, β, α) distribution, then the random variable $Y = X/\delta$ has MOEFR($1, \beta, \alpha$) distribution. Thus, we will consider the n^{th} moment of the random variable Y with MOEFR($1, \beta, \alpha$) distribution. Making the substitution $u = y^{-\beta}$, the n^{th} moment of the random variable Y can be written as

$$E(Y^n) = \alpha\beta \int_0^\infty \frac{y^{n-\beta-1}e^{-y^{-\beta}}}{(\alpha + (1-\alpha)e^{-y^{-\beta}})^2} dy = \alpha \int_0^\infty \frac{u^{-n/\beta}e^{-u}du}{(\alpha + (1-\alpha)e^{-u})^2}.$$

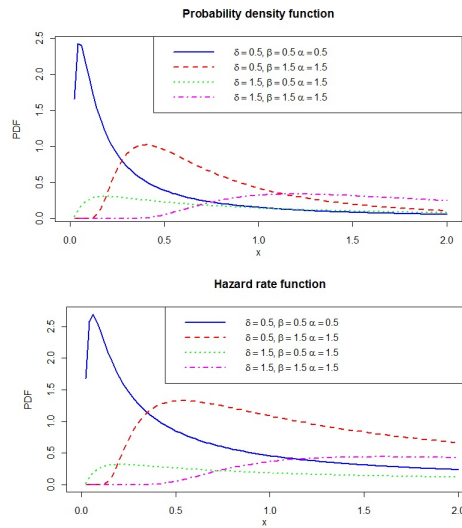


Figure 3.1: Probability density function (top) and hazard rate function (bottom) of the MOEFR distribution for different values of the parameters δ , β and α .

If $\alpha > 1/2$, then using the expansion

$$\frac{1}{(\alpha + (1 - \alpha)e^{-u})^2} = \frac{1}{\alpha^2} \sum_{i=1}^{\infty} i \left(\frac{\alpha - 1}{\alpha} \right)^{i-1} e^{-(i-1)u},$$

we obtain that for $n < \beta$ and $\alpha > 1/2$

$$\begin{aligned} E(Y^n) &= \frac{1}{\alpha} \sum_{i=1}^{\infty} \left(\frac{\alpha - 1}{\alpha} \right)^{i-1} i^{n/\beta} \Gamma\left(1 - \frac{n}{\beta}\right) \\ &= \alpha^{-1} \Gamma\left(1 - \frac{n}{\beta}\right) \Phi\left(\frac{\alpha - 1}{\alpha}, -\frac{n}{\beta}, 1\right), \end{aligned}$$

where Φ is Lerch transcendental function defined as $\Phi(x, s, \nu) = \sum_{n=0}^{\infty} \frac{x^n}{(n+\nu)^s}$. If $0 < \alpha < 2$, then using the expansion

$$\begin{aligned} \frac{1}{(\alpha + (1 - \alpha)e^{-u})^2} &= \sum_{i=1}^{\infty} i(1 - \alpha)^{i-1} (1 - e^{-u})^{i-1} \\ &= \sum_{i=1}^{\infty} i(1 - \alpha)^{i-1} \sum_{j=0}^{i-1} \binom{i-1}{j} (-1)^j e^{-ju}, \end{aligned}$$

we obtain that for $n < \beta$ and $0 < \alpha < 2$, the n^{th} moment is given as

$$E(Y^n) = \alpha \Gamma \left(1 - \frac{n}{\beta} \right) \sum_{i=1}^{\infty} i(1 - \alpha)^{i-1} \sum_{j=0}^{i-1} \binom{i-1}{j} (-1)^j (1 + j)^{n/\beta - 1}.$$

Figure 3.2 gives the expectation, standard deviation, skewness and kurtosis of random variable with MOEFR($1, \beta, \alpha$) distribution as a function of α and β . The q^{th} quantile of the

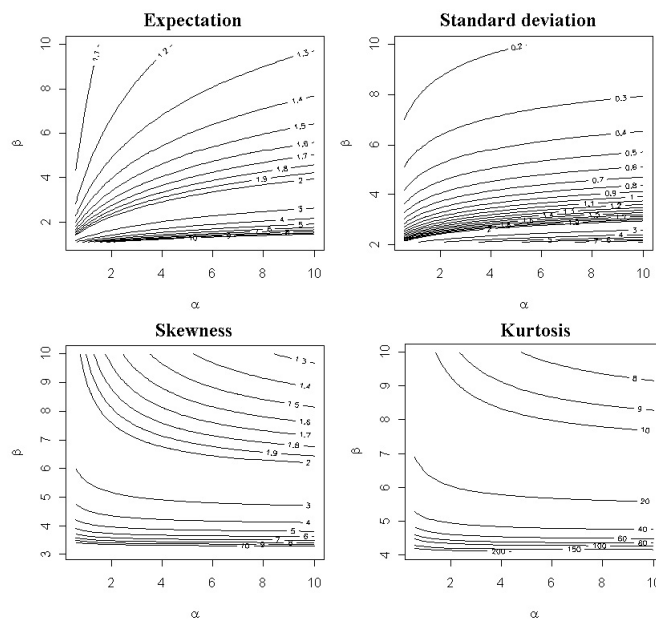


Figure 3.2: The expectation, standard deviation, skewness and kurtosis of a random variable with MOEFR($1, \beta, \alpha$) distribution.

MOEFR distribution is given by

$$x_q = G^{-1}(q) = \delta \left[\log \left(\frac{1 - (1 - \alpha)q}{\alpha q} \right) \right]^{-1/\beta}, \quad 0 \leq q \leq 1,$$

where $G^{-1}(\cdot)$ is the inverse distribution function. The median of the distribution is hence

$$\text{median}(X) = \delta \left[\log \left(\frac{\alpha + 1}{\alpha} \right) \right]^{-1/\beta}.$$

Rényi entropy The entropy represents a measure of uncertainty variation of a random variable. Rényi(1961) introduced a new measure of entropy called Rényi entropy. The Rényi entropy is defined as $I_R(\gamma) = \frac{1}{1-\gamma} \log \int_R g^\gamma(x) dx$, $\gamma > 0$ and $\gamma \neq 1$. Rényi entropy of order 1 is Shannon entropy. We consider first $g^\gamma(x)$ given by,

$$g^\gamma(x) = \frac{\alpha^\gamma \beta^\gamma \left(\frac{\gamma}{x}\right)^{(\beta+1)\gamma} e^{-\gamma(\delta/x)^\beta}}{\delta^\gamma \left(\alpha + (1 - \alpha)e^{-(\delta/x)^\beta}\right)^{2\gamma}}.$$

Suppose that $\alpha > \frac{1}{2}$. Using the series expansion

$$\left(\alpha + (1 - \alpha)e^{-(\delta/x)^\beta}\right)^{-2\gamma} = \alpha^{-2\gamma} \sum_{k=0}^{\infty} \frac{\Gamma(2\gamma + k)}{\Gamma(2\gamma)k!} \left(1 - \frac{1}{\alpha}\right)^k e^{-k(\delta/x)^\beta},$$

and by making the substitution $y = (\delta/x)^\beta$, for $(\beta + 1)\gamma > 1$ we have ,

$$\begin{aligned} \int_0^\infty g^\gamma(x) dx &= \frac{\beta^\gamma}{\alpha^\gamma \delta^\gamma} \sum_{k=0}^{\infty} \frac{\Gamma(2\gamma + k)}{\Gamma(2\gamma)k!} \left(1 - \frac{1}{\alpha}\right)^k \int_0^\infty \left(\frac{\delta}{x}\right)^{(\beta+1)\gamma} e^{-(k+\gamma)(\delta/x)^\beta} dx \\ &= \frac{\beta^{\gamma-1}}{\alpha^\gamma \delta^{\gamma-1}} \sum_{k=0}^{\infty} \frac{\Gamma(2\gamma + k)}{\Gamma(2\gamma)k!} \left(1 - \frac{1}{\alpha}\right)^k \Gamma\left(\frac{(\beta + 1)\gamma - 1}{\beta}\right) (k + \gamma)^{\frac{1 - (\beta+1)\gamma}{\beta}}. \end{aligned}$$

Thus we obtain that in the case $\alpha > 1/2$, the Rényi entropy is

$$I_R(\gamma) = \frac{1}{1 - \gamma} \log \left\{ \frac{\beta^{\gamma-1}}{\alpha^\gamma \delta^{\gamma-1}} \sum_{k=0}^{\infty} \frac{\Gamma(2\gamma + k)}{\Gamma(2\gamma)k!} \left(1 - \frac{1}{\alpha}\right)^k \Gamma\left(\frac{(\beta + 1)\gamma - 1}{\beta}\right) (k + \gamma)^{\frac{1 - (\beta+1)\gamma}{\beta}} \right\}.$$

Similarly, we can show that in the case $0 < \alpha < 2$ and by using the series expansion

$$\left(\alpha + (1 - \alpha)e^{-(\delta/x)^\beta}\right)^{-2\gamma} = \sum_{k=0}^{\infty} \frac{\Gamma(2\gamma + k)}{\Gamma(2\gamma)k!} (1 - \alpha)^k \left(1 - e^{-(\delta/x)^\beta}\right)^k,$$

the Rényi entropy is

$$I_R(\gamma) = \frac{1}{1 - \gamma} \log \left\{ \frac{\alpha^\gamma \beta^{\gamma-1}}{\delta^{\gamma-1}} \sum_{k=0}^{\infty} \frac{\Gamma(2\gamma + k)}{\Gamma(2\gamma)k!} (1 - \alpha)^k \sum_{j=0}^k \binom{k}{j} (-1)^j \times \Gamma\left(\frac{(\beta + 1)\gamma - 1}{\beta}\right) (j + \gamma)^{\frac{1 - (\beta + 1)\gamma}{\beta}} \right\}.$$

3.2.4 Compounding

Ghitany et al. (2005, 2007) and Ghitany and Kotz (2007) expressed Marshall–Olkin extended forms of Pareto, Weibull, Lomax and linear exponential family of distributions as a compound distribution with exponential distribution as mixing density. Now we show that under suitable conditions MOEFR distribution can be expressed as a compound distribution with exponential distribution as mixing density.

Theorem 3.2.3. *Let X be a continuous random variable with conditional pdf given as*

$$\bar{G}(x|\theta) = \exp \left\{ - \left(\left(1 - e^{-(\delta/x)^\beta} \right)^{-1} - 1 \right) \theta \right\}, \quad x, \delta, \beta, \theta > 0.$$

Let Θ follows an exponential distribution with pdf given by $m(\theta) = \alpha e^{-\alpha\theta}$, $\theta, \alpha > 0$.

Then the compound distribution of X becomes the MOEFR (δ, β, α) distribution.

Proof: For all $x > 0, \delta > 0, \beta > 0, \alpha > 0$, the unconditional survival function of X is given

by

$$\begin{aligned}\bar{G}(x) &= \int_0^\infty \bar{G}(x|\theta)m(\theta)d\theta \\ &= \alpha \int_0^\infty \exp \left\{ - \left(\left(1 - e^{-(\delta/x)^\beta} \right)^{-1} - 1 \right) \theta \right\} e^{-\alpha\theta} d\theta \\ &= \frac{\alpha(1 - e^{-(\delta/x)^\beta})}{1 - (1 - \alpha)(1 - e^{-(\delta/x)^\beta})},\end{aligned}$$

which is the survival function of a random variable with MOEFR distribution.

3.2.5 Order statistics

There are several books that deal with the asymptotic theory of extremes and their statistical applications. David(1981) and Arnold, Balakrishna and Nagarajha(1992) provide a compact account of the asymptotic theory of extremes. Galambos(1978,1987) present elaborate treatment of this topic. Let X_1, X_2, \dots, X_n be random variables from a population with the MOEFR(δ, β, α) distribution. Let $X_{i:n}$ denote the i^{th} order statistics. Then the probability density function of the random variable $X_{i:n}$ is

$$\begin{aligned}g_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!} g(x) G^{i-1}(x) (1 - G(x))^{n-i} \\ &= \frac{n!}{(i-1)!(n-i)!} \frac{\alpha^{n+1-i} \beta \left(\frac{\delta}{x}\right)^{\beta+1} e^{-i(\delta/x)^\beta} \left(1 - e^{-(\delta/x)^\beta}\right)^{n-i}}{\delta \left(\alpha + (1 - \alpha)e^{-(\delta/x)^\beta}\right)^{n+1}}.\end{aligned}$$

Now we will show that the probability density function $g_{i:n}(x)$ can be represented as infinite mixture of the beta Fréchet distributed random variables. As before, let us consider two cases, when $\alpha > 1/2$ and $0 < \alpha < 2$. Let us first consider the case $\alpha > 1/2$. Using the series expansion

$$\left(\alpha + (1 - \alpha)e^{-(\delta/x)^\beta}\right)^{-n-1} = \alpha^{-n-1} \sum_{k=0}^{\infty} \binom{n+k}{k} \left(1 - \frac{1}{\alpha}\right)^k e^{-k(\delta/x)^\beta},$$

we obtain that

$$\begin{aligned} g_{i:n}(x) &= \frac{n!\beta}{(i-1)!(n-i)!\alpha^i\delta} \sum_{k=0}^{\infty} \binom{n+k}{k} \left(1 - \frac{1}{\alpha}\right)^k \left(\frac{\delta}{x}\right)^{\beta+1} e^{-(k+i)(\delta/x)^\beta} (1 - e^{-(\delta/x)^\beta})^{n-i} \\ &= \frac{n!\alpha^{-i}}{(i-1)!(n-i)!} \sum_{k=0}^{\infty} \binom{n+k}{k} \left(1 - \frac{1}{\alpha}\right)^k B(k+i, n-i+1) \times \\ &\quad f_{BF}(x; \delta, \beta, k+i, n-i+1), \end{aligned}$$

where $f_{BF}(x; \delta, \beta, a, b)$ represents the beta Fréchet distribution with parameters δ , β , a and b . Similarly, in the case $0 < \alpha < 2$, using the series expansion

$$\left(\alpha + (1-\alpha)e^{-(\delta/x)^\beta}\right)^{-n-1} = \sum_{k=0}^{\infty} \binom{n+k}{k} (1-\alpha)^k (1 - e^{-(\delta/x)^\beta})^k,$$

we obtain that

$$\begin{aligned} g_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!} \frac{\alpha^{n-i+1}\beta}{\delta} \sum_{k=0}^{\infty} \binom{n+k}{k} (1-\alpha)^k \left(\frac{\delta}{x}\right)^{\beta+1} e^{-i(\delta/x)^\beta} (1 - e^{-(\delta/x)^\beta})^{n-i+k} \\ &= \frac{n!\alpha^{n-i+1}}{(i-1)!(n-i)!} \sum_{k=0}^{\infty} \binom{n+k}{k} \left(1 - \frac{1}{\alpha}\right)^k B(i, n-i+k+1) \times \\ &\quad f_{BF}(x; \delta, \beta, i, n-i+k+1). \end{aligned}$$

In some cases it is important to derive the asymptotic distribution of the sample maxima $X_{n:n} = \max(X_1, X_2, \dots, X_n)$. Its asymptotic distribution follows from the following theorem.

Theorem 3.2.4. *Let X_1, X_2, \dots, X_n be a random sample of size n from a population with the MOEFR distribution. Then*

$$\lim_{n \rightarrow \infty} P(X_{n:n} \leq b_n t) = \exp(-t^{-\beta}), \quad t > 0,$$

which is of Fréchet type, where

$$b_n = \delta \left[\log \left(1 + \frac{1}{\alpha(n-1)} \right) \right]^{\frac{-1}{\beta}}.$$

Proof: Since $\lim_{t \rightarrow \infty} t h(t) = \beta$, the proof follows from the Theorem 8.3.3 (Arnold et al., 1992).

3.2.6 Estimation of parameters

Since the moments of a random variable with the MOEFR distribution can not be presented in closed form, we must consider other methods for estimation of the unknown parameters. We estimate the unknown parameters of the Marshall-Olkin Extended Fréchet distribution using the maximum likelihood estimation method. Let x_1, x_2, \dots, x_n be an observed sample. Then the corresponding log-likelihood function is

$$\begin{aligned} \log L = & n \log \alpha + n \log \beta + (\beta + 1) \sum_{i=1}^n \log(\delta/x_i) - \sum_{i=1}^n (\delta/x_i)^\beta \\ & - n \log \delta - 2 \sum_{i=1}^n \log(\alpha + (1 - \alpha)e^{-(\delta/x_i)^\beta}). \end{aligned}$$

The normal equations become

$$\begin{aligned}\frac{\partial \log L}{\partial \delta} &= \frac{n(\beta + 1)}{\delta} - \beta \delta^{\beta-1} \sum_{i=1}^n x_i^{-\beta} - \frac{n}{\delta} \\ &\quad + 2(1 - \alpha) \beta \delta^{\beta-1} \sum_{i=1}^n \frac{e^{-(\delta/x_i)^\beta}}{x_i^\beta (\alpha + (1 - \alpha)e^{-(\delta/x_i)^\beta})} = 0 \\ \frac{\partial \log L}{\partial \beta} &= \frac{n}{\beta} + \sum_{i=1}^n \log(\delta/x_i) - \delta^\beta \sum_{i=1}^n x_i^{-\beta} \log(\delta/x_i) \\ &\quad + 2(1 - \alpha) \delta^\beta \sum_{i=1}^n \frac{e^{-(\delta/x_i)^\beta} \log(\delta/x_i)}{x_i^\beta (\alpha + (1 - \alpha)e^{-(\delta/x_i)^\beta})} = 0 \\ \frac{\partial \log L}{\partial \alpha} &= \frac{n}{\alpha} - 2 \sum_{i=1}^n \frac{1 - e^{-(\delta/x_i)^\beta}}{\alpha + (1 - \alpha)e^{-(\delta/x_i)^\beta}} = 0.\end{aligned}$$

Solution of these non-linear system of equations gives the estimates of the parameters. The maximum likelihood estimates can be easily obtained in any statistical package, for example in R using the function `nlm`.

3.3 Data analysis

In this section we compare our distribution with the Fréchet, the exponentiated Fréchet and beta Fréchet distributions. We consider the data set from Bjerkedal (1960). This data set consists 72 observations of survival times of injected guinea pigs with different doses of tubercle bacilli: 12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 60, 61, 62, 63, 65, 65, 67, 68, 70, 70, 72, 73, 75, 76, 76, 81, 83, 84, 85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 341, 376.

We estimate the unknown parameters of each distribution by the method of maximum likelihood estimation. Then for obtained maximum likelihood estimates we derived the values of two criterias AIC and BIC, the values of Kolmogorov-Smirnov statistics and the

Table 3.1: Maximum likelihood estimates, Kolmogorov-Smirnov statistics and p -values for the survival times of injected guinea pigs.

Distribution	Estimates				AIC	BIC	K-S	p -value
Fréchet(δ, β)	54.1888	1.4148			795.3	799.9	0.1520	0.0644
EF(δ, β, α)	336.3775	0.6207	8.2725		786.5	793.3	0.1003	0.4358
BF(δ, β, a, b)	33.1871	0.1805	47.7665	65.1072	788.5	797.6	0.0998	0.4418
MOF(δ, β, α)	14.2086	2.4810	61.7630		785.4	792.2	0.0896	0.5788

appropriate p -values. The results are presented in Table 3.1.

Also, we draw the estimated cumulative distribution functions and the P-P plots for each fitted distributions and are presented in Figure 3.3 and in Figur3.4. We can see that the Marshall-Olkin Extended Fréchet distribution provides a good fit and can be used as a competitive model to the other considered models.

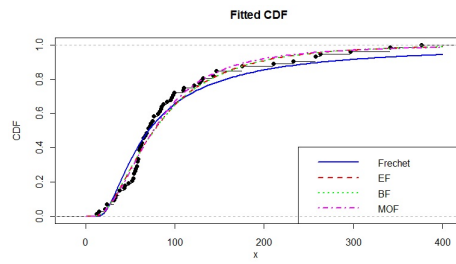


Figure 3.3: The fitted CDF for the data from Bjerkedal (1960).

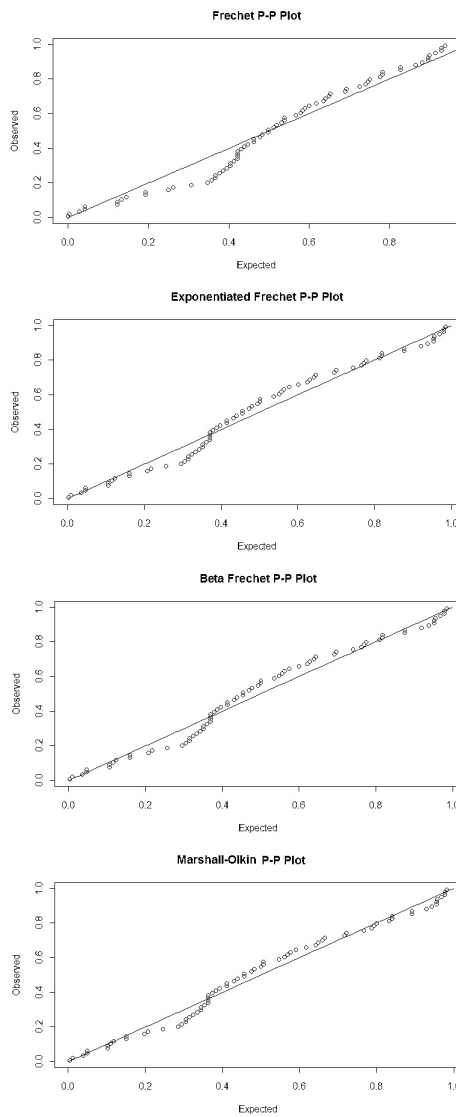


Figure 3.4: The P-P plots for the data from Bjerkedal (1960).

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Applications of Marshall-Olkin Extended Fréchet Distribution

4.1 Introduction

In chapter 3 we have introduced Marshall-Olkin Fréchet distribution with survival function

$$\bar{G}(x) = \frac{\alpha[1 - e^{-(\delta/x)^\beta}]}{\alpha + (1 - \alpha)e^{-(\delta/x)^\beta}}, \quad x, \delta, \beta, \alpha > 0. \quad (4.1.1)$$

and studied various properties including estimation of parameters. As a sequel, in this chapter we discuss the applications of the newly developed distribution in reliability contexts, acceptance sampling and time series analysis. The reliability is defined as the probability of not failing denoted by $R = P(X < Y)$ where X represents the stress and Y represents the the strength of a component. This measure of reliability is widely used in engineering problems. It may be noted that R has more interest than just a reliability measure. It can be used as a general measure of difference between two populations such as

treatment group and control group in bio-statistical contexts and clinical trials. In most of the work in the evaluation of $R = P(X < Y)$ it is assumed that both random variables has distribution belonging to the same family and more significantly it assumes independence between them. This problem has been extensively studied for various probability models including exponential, generalised exponential, gamma, Weibull and Burr distribution etc. For details one can refer the works of Constantine et al (1986) for the gamma case, Surles and Padgett (2001) for Burr type X model, Nadarajah (2004) for Laplace distribution, Kundu and Gupta (2005) for generalised exponential distribution, Kundu and Gupta (2006) for Weibull distribution, Kakade et al (2008) for exponentiated Gumbel distribution, Raqab et al (2008) for 3-parameter generalised exponential distribution, Gupta et al (2010) for Marshall-Olkin extended Lomax distribution and recently Bindu (2011) for double Lomax distribution. Raqab and Kundu (2005) made study on the comparison of different estimators of $P(Y < X)$ for a scaled Burr Type X distribution. Acceptance sampling plans are hypothesis tests of the product that has been submitted for an appraisal and subsequently resulted with acceptance or rejection. A sample is selected or checked for various characteristics. The decision is based on the amount of defect or defective units found in the sample. Effective sampling techniques involves effective selection of the products and the application of specific rules for lot inspection that follows the standards. In literature various authors have developed acceptance sampling plan based on different probability models. For example one can refer the works of Kantam and Rosaiah (1998) based on Half logistic model, Kantam et al (2001) based on Log-logistic model, Rosaiah and Kantam (2005) based on Inverse Rayleigh model, Rosaiah et al (2006) based on exponentiated Log-logistic model and Srinivasa Rao et al (2009) based on Marshall-Olkin extended exponential model. Time series modeling is finding its application in diversified fields today. Economics, social sciences, demography, medical sciences, actuarial science are very few of them. Warming trend in global temperature, levels of pollution causing mortality in a particular region are other major areas in present scenario where time series modeling is found effective. Gaver and Lewis (1980) developed a first order autoregressive time series model with exponential

stationary marginal distribution. They extended it to the case of gamma and mixed exponential processes. Jayakumar and Pillai (1993) extended it to the case of Mittag–Leffler processes. Several authors have developed similar processes with other non-Gaussian marginals like Weibull, Laplace, Linnik etc. Brown et al. (1984), Gibson (1986), Anderson and Arnold (1993), Alice and Jose (2001,2004), Naik and Jose (2008) are some of the researchers who worked on this topic.

In this chapter we discuss stress-strength analysis with respect to a simulated data as well as a real data based on Marshall-Olkin extended Fréchet distribution. We develop an acceptance sampling plan for the life time of a product following the new distribution. Finally we develop four types of AR(1) models with the extended distribution as marginals and derive some properties of these models.

4.2 Stress-strength analysis

Here we consider the statistical inference of the stress-strength parameter

$R = P(X < Y)$ when X and Y are independent Marshall-Olkin extended Fréchet random variables with parameters $(\delta, \beta, \alpha_1)$ and $(\delta, \beta, \alpha_2)$, respectively. Then using Gupta et al(2010), we obtain ,

$$\begin{aligned} P(X < Y) &= \int_{-\infty}^{\infty} P(Y > X/X = x)g_X(x)dx \\ &= \int_0^{\infty} \frac{\alpha_2(1 - e^{-(\frac{\delta}{x})^\beta})}{\alpha_2 + (1 - \alpha_2)e^{-(\frac{\delta}{x})^\beta}} \frac{\alpha_1\beta\delta^\beta e^{-(\frac{\delta}{x})^\beta}}{x^{\beta+1}(\alpha_1 + (1 - \alpha_1)e^{-(\frac{\delta}{x})^\beta})^2} dx \\ &= \frac{\alpha}{(\alpha - 1)^2} [-\log \alpha + \alpha - 1], \end{aligned}$$

where $\alpha = \alpha_2/\alpha_1$.

Now we consider the pdf of the Marshall-Olkin Extended Fréchet distribution given by

$$g(x; \alpha, \delta, \beta) = \frac{\alpha\beta\delta^\beta e^{-(\frac{\delta}{x})^\beta}}{x^{\beta+1}(\alpha + (1 - \alpha)e^{-(\frac{\delta}{x})^\beta})^2}, \quad x, \alpha, \delta, \beta > 0.$$

Let (x_1, \dots, x_m) and (y_1, \dots, y_n) be two independent random samples of sizes m and n from Marshall-Olkin Extended Fréchet distributions with tilt parameters α_1 and α_2 , respectively, and common unknown parameters δ and β . The log likelihood function is given by

$$\begin{aligned} L(\alpha_1, \alpha_2, \delta, \beta) &= \sum_{i=1}^m \log g(x_i; \alpha_1, \delta, \beta) + \sum_{i=1}^n \log g(y_i; \alpha_2, \delta, \beta) \\ &= m \log \alpha_1 + n \log \alpha_2 + (m+n) \log \beta + (m+n)\beta \log \delta - \sum_{i=1}^m (\delta/x_i)^\beta \\ &\quad - \sum_{j=1}^m (\delta/y_j)^\beta - (\beta+1) \sum_{i=1}^m \log x_i - (\beta+1) \sum_{j=1}^n \log y_j \\ &\quad - 2 \sum_{i=1}^m \log(\alpha_1 + (1-\alpha_1)e^{-(\delta/x_i)^\beta}) - 2 \sum_{j=1}^n \log(\alpha_2 + (1-\alpha_2)e^{-(\delta/y_j)^\beta}). \end{aligned}$$

The maximum likelihood estimates of the unknown parameters α_1, α_2 are the solutions of the non-linear equations

$$\begin{aligned} \frac{\partial L}{\partial \alpha_1} &= \frac{m}{\alpha_1} - 2 \sum_{i=1}^m \frac{1 - e^{-(\frac{\delta}{x_i})^\beta}}{\alpha_1 + (1-\alpha_1)e^{-(\frac{\delta}{x_i})^\beta}} = 0, \\ \frac{\partial L}{\partial \alpha_2} &= \frac{n}{\alpha_2} - 2 \sum_{j=1}^n \frac{1 - e^{-(\frac{\delta}{y_j})^\beta}}{\alpha_2 + (1-\alpha_2)e^{-(\frac{\delta}{y_j})^\beta}} = 0. \end{aligned}$$

The elements of information matrix are

$$\begin{aligned}
 I_{11} &= -E \left(\frac{\partial^2 L}{\partial \alpha_1^2} \right) \\
 &= \frac{m}{\alpha_1^2} - 2mE \left(\frac{(1 - e^{-\left(\frac{\delta}{x}\right)^\beta})^2}{[1 - \bar{\alpha}_1(1 - e^{-\left(\frac{\delta}{x}\right)^\beta})]^2} \right) \\
 &= \frac{m}{\alpha_1^2} - 2m \int_0^\infty \frac{(1 - e^{-\left(\frac{\delta}{x}\right)^\beta})^2 \alpha_1 \beta \left(\frac{\delta}{x}\right)^{\beta+1} e^{-\left(\frac{\delta}{x}\right)^\beta} dx}{\delta(1 - \bar{\alpha}_1(1 - e^{-\left(\frac{\delta}{x}\right)^\beta})^4} \\
 &= \frac{m}{\alpha_1^2} - 2m\alpha_1 \int_0^1 \frac{t^2}{(1 - \bar{\alpha}t)^4} dt \\
 &= m \left(\frac{1}{\alpha_1^2} - \frac{2}{3\alpha_1^2} \right) \\
 &= \frac{m}{3\alpha_1^2}.
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 I_{22} &= -E \left(\frac{\partial^2 L}{\partial \alpha_2^2} \right) = \frac{n}{3\alpha_2^2} \\
 I_{12} = I_{21} &= -E \left(\frac{\partial^2 L}{\partial \alpha_1 \partial \alpha_2} \right) = 0.
 \end{aligned}$$

By the property of m.l.e for $m \rightarrow \infty, n \rightarrow \infty$, we obtain that

$$(\sqrt{m}(\hat{\alpha}_1 - \alpha_1), \sqrt{n}(\hat{\alpha}_2 - \alpha_2))^T \xrightarrow{d} N_2(\mathbf{0}, \text{diag}\{a_{11}^{-1}, a_{22}^{-1}\}),$$

where $a_{11} = \lim_{m,n \rightarrow \infty} \frac{1}{m} I_{11} = \frac{1}{3\alpha_1^2}$ and $a_{22} = \lim_{m,n \rightarrow \infty} \frac{1}{n} I_{22} = \frac{1}{3\alpha_2^2}$. The 95% confidence interval for R is given by

$$\hat{R} \mp 1.96 \hat{\alpha}_1 b_1(\hat{\alpha}_1, \hat{\alpha}_2) \sqrt{\frac{3}{m} + \frac{3}{n}},$$

where $\hat{R} = R(\hat{\alpha}_1, \hat{\alpha}_2)$ is the estimator of R and

$$b_1(\alpha_1, \alpha_2) = \frac{\partial R}{\partial \alpha_1} = \frac{\alpha_2}{(\alpha_1 - \alpha_2)^3} \left[2(\alpha_1 - \alpha_2) + (\alpha_1 + \alpha_2) \log \frac{\alpha_2}{\alpha_1} \right].$$

4.2.1 Simulation Study

We generate $N = 10,000$ sets of X -samples and Y -samples from Marshall-Olkin Extended Fréchet distribution with parameters α_1, δ, β and α_2, δ, β , respectively. The combinations of samples of sizes $m = 20, 25, 30$ and $n = 20, 25, 30$ are considered. The estimates of α_1 and α_2 are then obtained from each sample to obtain \hat{R} . The validity of the estimate of R is discussed by the measures:

- 1) Average bias of the simulated N estimates of R :

$$\frac{1}{N} \sum_{i=1}^N (\hat{R}_i - R)$$

- 2) Average mean square error of the simulated N estimates of R :

$$\frac{1}{N} \sum_{i=1}^N (\hat{R}_i - R)^2$$

- 3) Average length of the asymptotic 95% confidence intervals of R :

$$\frac{1}{N} \sum_{i=1}^N 2(1.96)\hat{\alpha}_{1i} b_{1i}(\hat{\alpha}_{1i}, \hat{\alpha}_{2i}) \sqrt{\frac{3}{m} + \frac{3}{n}}$$

- 4) The coverage probability of the N simulated confidence intervals given by the proportion of such interval that include the parameter R .

The numerical values obtained for the measures listed above are presented in Tables 4.1 and 4.2. For $\alpha_1 < \alpha_2$ the average bias is positive and for $\alpha_1 > \alpha_2$, the average bias is negative but in both cases the average bias decreases as the sample size increases. The performance of confidence interval is quite satisfactory. The coverage probability is close to 0.95 and approaches the nominal value as the sample size increases. The simulation study indicates that the average bias, average MSE, average confidence interval and coverage probability do not show much variability for various parameter combinations.

Data analysis Let us consider now the data from Gupta et al (2010). We consider two

Table 4.1: Average bias and average MSE of the simulated estimates of R for $\delta = 3$ and $\beta = 2$.

(α_1, α_2)								
(m, n)	Average bias (b)				Average Mean Square Error AMSE			
	(0.5,0.8)	(0.8,1.2)	(0.8,0.5)	(1.2,0.8)	(0.5,0.8)	(0.8,1.2)	(0.8,0.5)	(1.2,0.8)
(20,20)	0.0833	0.0739	-0.0830	-0.0732	0.0083	0.0071	0.0084	0.0070
(20,25)	0.0830	0.0736	-0.0836	-0.0742	0.0085	0.0071	0.0085	0.0072
(20,30)	0.0820	0.0740	-0.0833	-0.0736	0.0084	0.0072	0.0084	0.0071
(25,20)	0.0851	0.0763	-0.0833	-0.0717	0.0086	0.0072	0.0079	0.0066
(25,25)	0.0844	0.0755	-0.0814	-0.0721	0.0085	0.0071	0.0079	0.0067
(25,30)	0.0846	0.0752	-0.0809	-0.0720	0.0085	0.0071	0.0079	0.0067
(30,20)	0.0862	0.0763	-0.0798	-0.0714	0.0087	0.0072	0.0076	0.0065
(30,25)	0.0859	0.0764	-0.7999	-0.0706	0.0087	0.0072	0.0076	0.0063
(30,30)	0.0852	0.0762	-0.0801	-0.0710	0.0085	0.0072	0.0077	0.0064

Table 4.2: Average confidence length and coverage probability of the simulated 95 percentage confidence intervals of R for $\delta = 3$ and $\beta = 2$.

(α_1, α_2)								
(m, n)	Average confidence length				coverage probability			
	(0.5,0.8)	(0.8,1.2)	(0.8,0.5)	(1.2,0.8)	(0.5,0.8)	(0.8,1.2)	(0.8,0.5)	(1.2,0.8)
(20,20)	0.3559	0.3557	0.3558	0.3556	0.989	0.9912	0.9894	0.9925
(20,25)	0.3376	0.334	0.3376	0.3373	0.9797	0.9848	0.9817	0.9865
(20,30)	0.3248	0.3246	0.3215	0.346	0.9755	0.979	0.9763	0.9801
(25,20)	0.3377	0.3376	0.338	0.3348	0.9816	0.9871	0.9889	0.9911
(25,25)	0.3183	0.3182	0.3185	0.3183	0.9715	0.9791	0.9799	0.9856
(25,30)	0.3049	0.3047	0.305	0.3048	0.9574	0.9718	0.9724	0.9774
(30,20)	0.3248	0.3249	0.3252	0.325	0.9789	0.9832	0.988	0.9916
(30,25)	0.305	0.3048	0.3055	0.3049	0.9625	0.9724	0.9799	0.9844
(30,30)	0.2908	0.2906	0.2909	0.2907	0.9512	0.9624	0.9665	0.9767

data sets which represents the times (in hours) of successive failure intervals of the air conditioning system of two jet planes.

The data set for X is 23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95.

The data set for Y is 487, 18, 100, 7, 98, 5, 85, 91, 43, 230, 3, 130. First we fit the Fréchet distribution with parameters δ and β for each data set separately. For the first data set we obtained the estimates $\hat{\delta} = 14.616$ and $\hat{\beta} = 0.724$ with the estimated log-

likelihood as -155.1144 . For the second data set we obtained the estimates $\hat{\delta} = 20.796$ and $\hat{\beta} = 0.656$ with the estimated log-likelihood as -69.2517 . Now we consider the values $\hat{\delta} = (14.616 + 20.796)/2 = 17.706$ and $\hat{\beta} = (0.724 + 0.656)/2 = 0.690$.

We test the null hypotheses that the true values are $\delta = 17.706$ and $\beta = 0.690$. The chi-square statistic and the p -value of the likelihood ratio test are 0.5098 and 0.7750 respectively.

For the second data set, the chi-square statistic and the p -value of the likelihood ratio test are respectively 0.1405 and 0.9322 . We can conclude that we can accept the null hypotheses that the true values are $\delta = 17.706$ and $\beta = 0.690$.

Table 4.3 gives a comparison between the Fréchet model and the Lomax model given in Gupta et al (2010). It is clear that the Fréchet model is a better fit than the other.

Table 4.3: χ^2 and p -values for the data set of successive failure times of the air conditioning system of two jet planes

Distribution	χ^2 value		p-Value	
	plane-1	plane-2	plane-1	plane-2
Lomax	1.0232	1.0194	0.5995	0.6007
Fréchet	0.5098	0.1405	0.7750	0.9322

Now we derive the estimates of the parameters α_1 and α_2 for the Marshall-Olkin Extended Fréchet distribution. We obtain $\hat{\alpha}_1 = 0.9501$, $\hat{\alpha}_2 = 1.1241$ and $\hat{R} = 0.5280$ with standard error $SE(\hat{R}) = 0.0983$. The asymptotic 95% confidence interval of R is $(0.3353, 0.7207)$.

4.3 Reliability test plan

Here we discuss the reliability test plan for accepting or rejecting a lot where the life time of the product follows Marshall-Olkin Extended Fréchet distribution. In a life testing experiment the procedure is to terminate the test by a pre-determined time ' t ' and note the number of failures. If the number of failures at the end of time ' t ' does not exceed a given number ' c ', called acceptance number then we accept the lot with a given probability of at least p^* . But if the number of failures exceeds ' c ' before time ' t ' then the test is terminated

and the lot is rejected. For such truncated life test and the associated decision rule we are interested in obtaining the smallest sample size to arrive at a decision. For Marshall-Olkin Extended Fréchet distribution with probability of failure,

$$G(x, \alpha, \beta, \delta) = \frac{e^{-(\frac{\delta}{x})^\beta}}{\alpha + \bar{\alpha}e^{-(\frac{\delta}{x})^\beta}}, \quad x, \alpha, \beta, \delta > 0, \quad (4.3.1)$$

the average life time depends only on δ if α and β are known. Let δ_0 be the required minimum average life time. Then

$$G(x, \alpha, \beta, \delta) \leq G(x, \alpha, \beta, \delta_0) \Leftrightarrow \delta \geq \delta_0.$$

A sampling plan is specified by the following quantities:

- 1) the number of units n on test,
- 2) the acceptance number c ,
- 3) the maximum test duration t , and
- 4) the minimum average lifetime represented by δ_0 .

The consumers risk, i.e. the probability of accepting a bad lot should not exceed the value $1 - p^*$, where p^* is a lower bound for the probability that a lot of true value δ below δ_0 is rejected by the sampling plan. For fixed p^* the sampling plan is characterized by $(n, c, t/\delta_0)$. By sufficiently large lots we can apply binomial distribution to find acceptance probability. The problem is to determine the smallest positive integer 'n' for given value of c and t/δ_0 such that

$$L(p_0) = \sum_{i=0}^c \binom{n}{i} p_0^i (1 - p_0)^{n-i} \leq 1 - p^*, \quad (4.3.2)$$

where $p_0 = G(t, \alpha, \beta, \delta_0)$. The function $L(p)$ is called operating characteristic function of the sampling plan, i.e. the acceptance probability of the lot as a function of the failure

probability $p(\delta) = G(t, \alpha, \beta, \delta)$. The average life time of the product is increasing with δ and therefore the failure probability $p(\delta)$ decreases implying that the operating characteristic function is increasing in δ . The minimum values of n satisfying (4.3.2) are obtained for $\alpha = 2, \beta = 2$ and $p^* = 0.75, 0.90, 0.95, 0.99$ and $t/\delta_0 = 0.90, 0.1.05, 1.4, 1.65, 1.90, 2.15, 2.4$ and 2.65 . The results are displayed in Table 4.4. If $p_0 = G(t, \alpha, \beta, \delta_0)$ is small and n is large, the binomial probability may be approximated by Poisson probability with parameter $\lambda = np_0$ so that (4.3.2) becomes

$$L_1(p_0) = \sum_{i=0}^c \frac{\lambda^i}{i!} e^{-\lambda} \leq 1 - p^* . \quad (4.3.3)$$

The minimum values of n satisfying (4.3.3) are obtained for the same combination of values of α, β and t/δ_0 for various values of p^* are presented in Table 4.5. The operating characteristic function of the sampling plan $(n, c, t/\delta_0)$ gives the probability $L(p)$ of accepting the lot with

$$L(p) = \sum_{i=0}^c \binom{n}{i} p_0^i (1 - p_0)^{n-i} \quad (4.3.4)$$

where $p = G(t, \delta)$ is considered as a function of δ . For given $p^*, t/\delta_0$ the choice of c and n are made on the basis of characteristics. Considering the fact that

$$p = G\left(\frac{t}{\delta_0} / \frac{\delta}{\delta_0}\right) \quad (4.3.5)$$

values of operating characteristics for a few sampling plans are calculated and presented in Table 4.6

The producers risk is the probability of rejecting a lot when $\delta > \delta_0$. For the given value of producers risk say say 0.05 we obtain p from the sampling plan given in Table 4.4 and satisfying the equation (4.3.5)

$$\sum_{i=0}^c \binom{n}{i} p_0^i (1 - p_0)^{n-i} \geq 0.95 \quad (4.3.6)$$

If one is interested in knowing what value of δ/δ_0 will ensure a producers risk less than or equal to 0.05 if a sampling plan under discussion is adopted it can be obtained from $p = G(t, \alpha, \beta, \delta)$ and then using (4.3.5) and is presented in Table 4.7.

4.3.1 Illustration of Table and application of sampling plan

Assume that the life time distribution is Marshall-Olkin Extended Fréchet distribution with $\alpha = 2$ and $\beta = 2$. Suppose that the experimenter is interested in establishing that the true unknown average lifetime is at least 952 hours. Suppose that it is desired to stop the the experiment at $t=1000$ hours. So if consumers risk is set to be $1 - p^* = .10$ then from Table 5.4 sampling plan is $(n = 20, c = 2, t/\delta_0 = 1.05)$. ie; if during 1000 hours, not more than 2 failures out of 20 are observed then the experimenter can assert with confidence limit 0.90 that the average life is at least 1050 hours. If we use Poisson approximation to binomial the corresponding value is $n=22$.

For the sampling plan $(n = 11, c = 2, t/\delta_0 = 1.4)$ with the consumer risk 0.10 under the Marshall-Olkin Extended Fréchet distribution the operating characteristic values from Table 4.6 are,

$\frac{\delta}{\delta_0}$	1.6	1.8	1.95	2.0	2.2	2.4	2.6
L(p)	0.7584	0.8975	0.9522	0.9637	0.9891	0.9972	0.9994

This shows that when $\frac{\delta}{\delta_0} = 1.95$ producers risk is 0.05 and when $\frac{\delta}{\delta_0} = 2.4$ it is negligible. From Table 4.6 for this plan it can be observed that the minimum of $\frac{\delta}{\delta_0}$ which gives the producers risk as 0.05 is 1.96.

Similarly, if consumers risk is fixed as 0.05 then from Table 4.6 for the sampling plan $(n = 13, c = 2, t/\delta_0 = 1.4)$ and with consumer risk is 0.05, the operating characteristic values are,

$\frac{\delta}{\delta_0}$	1.5	2	2.03	2.5	3	3.5	4
L(p)	0.5443	0.9434	0.9519	0.9977	0.9999	1.0000	1.0000

This describes that when $\frac{\delta}{\delta_0} = 2.03$ producers risk is 0.05 and when $\frac{\delta}{\delta_0} = 2.5$ it is negligible. From Table 4.7 for this plan it can be observed that the minimum of $\frac{\delta}{\delta_0}$ which gives the producers risk as 0.05 is also 2.03.

It is clear from the above descriptions that if the consumer risk is fixed at a particular level the quality improvement may be required only up to a fixed extended level.

The OC curve drawn for $p^* = 0.75$ and for the sampling plan $(n, c, \frac{t}{\delta_0})$ (Fig 4.1) indicates the discriminating power of the sampling plan at various values of $\frac{t}{\delta_0}$, giving the quality realised by the product of specific producers risk. Using graphs drawn for $(p^*, \frac{t}{\delta_0})$ and for various values of c , (Fig.4. 2, Fig.4. 3 and Fig.4.4) one can obtain the minimum ratio of $\frac{\delta}{\delta_0}$ at any specified level giving minimum quality realised by the selected process.

Application Consider a simulated data of failure times generated from Marshall-olkin Extended Fréchet distribution with $\alpha = 2, \beta = 2$ and $\delta = 4$ of size 14. The ordered sample is 482,1624,1707,1855,2135,2194,2207,2368,

2812,3061,3164,3784,4958,5383

Assume that the required average is 1000 hours and the testing time is 1050 hours. This leads to the ratio $t/\delta_0 = 1.05$ From Table 1 the sampling plan for $p^* = 0.90$ is

$(n = 14, c = 1, t/\delta_0 = 1.05)$. We accept the lot only if the number of failures before 1050 hours is less than or equal to 1. In the above sample there is only one failure at 482 hours before termination $t=1050$ hours. Hence we accept the product. If one is interested in knowing what value of δ/δ_0 will ensure a producers risk less than or equal to 0.05 if a sampling plan under discussion is adopted it can be obtained from $p = G(t, \delta)$ and then using (4.3.5) and is presented in Table 6. The minimum values of n satisfying (4.3.3) are obtained for the same combination of values of α, β and t/δ_0 for various values of p^* are presented in Table 4.

Table 4.4: Minimum sample size for the specified ratio t/δ_0 , confidence level p^* , acceptance number c , $\alpha = 2$ and $\beta = 2$ using binomial approximation.

p^*	c	t/δ_0							
		0.9	1.05	1.4	1.65	1.9	2.15	2.4	2.65
0.75	0	8	5	3	2	2	2	2	1
	1	15	10	6	5	4	3	3	3
	2	23	15	9	7	6	5	5	4
	3	29	20	11	9	8	7	6	6
	4	36	24	14	11	9	8	8	7
	5	43	29	16	13	11	10	9	9
	6	49	33	19	15	13	12	11	10
	7	56	37	22	17	15	13	12	11
	8	63	42	24	19	17	15	14	13
	9	69	46	27	21	18	16	15	14
10	75	50	29	23	20	18	17	16	
0.90	0	13	8	5	4	3	3	2	2
	1	22	14	8	6	5	5	4	4
	2	30	20	11	9	7	6	6	5
	3	38	25	14	11	9	8	7	7
	4	45	30	17	13	11	10	9	8
	5	53	35	20	16	13	12	11	10
	6	60	40	23	18	15	13	12	11
	7	67	45	25	20	17	15	14	13
	8	74	49	28	22	19	17	15	14
	9	81	54	31	24	21	18	17	16
10	88	59	33	26	23	20	18	17	
0.95	0	17	11	6	4	4	3	3	3
	1	26	17	10	7	6	5	5	4
	2	35	23	13	10	8	7	7	6
	3	44	29	16	12	10	9	8	8
	4	52	34	19	15	12	11	10	9
	5	59	39	22	17	14	13	12	11
	6	67	44	25	20	16	15	13	12
	7	75	49	28	22	18	16	15	14
	8	82	54	31	24	20	18	16	15
	9	89	59	33	26	22	20	18	17
10	97	64	36	29	24	21	20	18	
0.99	0	23	16	9	7	5	5	4	4
	1	34	24	13	10	8	7	6	6
	2	43	30	17	13	11	9	8	7
	3	52	37	20	15	13	11	10	9
	4	60	42	23	18	15	13	12	11
	5	68	48	27	21	17	15	14	12
	6	75	54	30	23	19	17	15	14
	7	83	59	33	26	21	19	17	16
	8	90	64	36	28	23	21	19	17
	9	98	70	39	30	25	22	20	19
10	105	75	42	33	27	24	22	20	

Table 4.5: Minimum sample size for the specified ratio t/δ_0 , confidence level p^* , acceptance number c , $\alpha = 2$ and $\beta = 2$ using Poisson approximation.

p^*	c	t/δ_0							
		0.9	1.05	1.4	1.65	1.9	2.15	2.4	2.65
0.75	0	8	6	4	3	3	3	2	2
	1	15	11	7	6	5	4	4	4
	2	22	16	10	8	7	6	6	6
	3	28	21	12	10	9	8	8	7
	4	35	25	15	12	11	10	9	9
	5	41	30	18	15	13	12	11	10
	6	47	34	20	17	15	13	12	12
	7	53	39	23	19	16	15	14	13
	8	59	43	26	21	18	17	15	15
	9	65	48	28	23	20	18	17	16
10	71	52	31	25	22	20	18	18	
0.9	0	13	10	6	5	4	4	4	4
	1	22	16	10	8	7	6	6	6
	2	30	22	13	11	9	8	8	7
	3	37	27	16	13	11	10	10	9
	4	44	32	19	16	14	12	12	11
	5	51	37	22	18	16	14	13	13
	6	58	42	25	20	18	16	15	14
	7	65	47	28	23	20	18	17	16
	8	72	52	31	25	22	20	18	17
	9	78	57	34	27	24	22	20	19
10	85	62	36	30	26	23	22	21	
0.95	0	17	12	7	6	5	5	5	4
	1	26	19	12	9	8	8	7	7
	2	35	25	15	12	11	10	9	9
	3	43	31	19	15	13	12	11	11
	4	50	37	22	18	15	14	13	12
	5	58	42	25	20	18	16	15	14
	6	65	47	28	23	20	18	17	16
	7	72	52	31	25	22	20	19	18
	8	79	58	34	28	24	22	20	19
	9	86	63	37	30	26	24	22	21
10	93	68	40	33	28	26	24	23	
0.99	0	26	19	11	9	8	7	7	7
	1	37	27	16	13	11	10	10	9
	2	46	34	20	16	14	13	12	11
	3	55	40	24	19	17	15	14	14
	4	64	46	28	22	20	18	17	16
	5	68	52	31	25	22	20	19	18
	6	75	58	34	28	24	22	21	20
	7	83	64	38	31	27	24	23	21
	8	90	69	41	33	29	26	25	23
	9	98	75	44	36	31	28	26	25
10	105	80	47	39	34	30	28	27	

Table 4.6: Values of the operating characteristic function of the sampling plan $(n, c, \frac{t}{\delta_0})$

p^*	n	c	$\frac{t}{\delta_0}$	$\frac{\delta}{\delta_0}$					
				1.6	1.8	2.0	2.2	2.4	2.6
0.75	23	2	0.90	0.9870	0.9988	0.9999	1.0000	1.0000	1.0000
	15	2	1.05	0.9608	0.9928	0.9990	0.9999	1.0000	1.0000
	9	2	1.40	0.8446	0.9389	0.9795	0.9941	0.9985	0.9997
	7	2	1.65	0.7716	0.8855	0.9487	0.9793	0.9924	0.9975
	6	2	1.90	0.6942	0.8210	0.9039	0.9522	0.9779	0.9905
	5	2	2.15	0.6769	0.7952	0.8783	0.9317	0.9637	0.9817
	5	2	2.40	0.5526	0.6853	0.7910	0.8682	0.9207	0.9544
	4	2	2.65	0.6380	0.7446	0.8271	0.8873	0.9290	0.9567
0.9	30	2	0.9	0.9733	0.9973	0.9998	1.0000	1.0000	1.0000
	20	2	1.05	0.9188	0.9838	0.9997	0.9977	1.0000	1.0000
	11	2	1.4	0.7584	0.8975	0.9637	0.9891	0.9972	0.9994
	9	2	1.65	0.6213	0.7911	0.8991	0.9568	0.9835	0.9943
	7	2	1.9	0.5868	0.7440	0.8558	0.9256	0.9646	0.9844
	6	2	2.15	0.5375	0.6871	0.8035	0.8847	0.9364	0.9669
	6	2	2.4	0.3979	0.5477	0.6815	0.7888	0.8675	0.9210
	5	2	2.65	0.4402	0.5741	0.6921	0.7875	0.8595	0.9107
p^*	n	c	$\frac{t}{\delta_0}$	$\frac{\delta}{\delta_0}$					
				1.5	2	2.5	3	3.5	4
0.95	35	2	0.9	0.8987	0.9997	1.0000	1.0000	1.0000	1.0000
	23	2	1.05	0.7878	0.9964	0.9999	1.0000	1.0000	1.0000
	13	2	1.4	0.5443	0.9434	0.9977	0.9999	1.0000	1.0000
	10	2	1.65	0.4375	0.8692	0.9865	0.9992	0.9999	1.0000
	8	2	1.9	0.3289	0.7213	0.9279	0.9874	1.0000	1.0000
	7	2	2.15	0.3289	0.7213	0.9279	0.9874	1.0000	1.0000
	7	2	2.4	0.2084	0.5719	0.8462	0.9609	0.9926	0.9989
	6	2	2.65	0.2237	0.5558	0.8170	0.9428	0.9859	0.9972
0.99	43	2	0.9	0.8408	0.9994	0.9999	1.0000	1.0000	1.0000
	30	2	1.05	0.6531	0.9924	0.9999	1.0000	1.0000	1.0000
	17	2	1.4	0.3524	0.8903	0.9949	0.9999	1.0000	1.0000
	13	2	1.65	0.2492	0.7662	0.9714	0.9983	1.0000	1.0000
	11	2	1.9	0.1703	0.6247	0.9164	0.9898	0.9992	1.0000
	9	2	2.15	0.2311	0.5543	0.8621	0.9732	1.0000	1.0000
	8	2	2.4	0.1302	0.46963	0.7896	0.9428	0.9887	0.9983
	7	2	2.65	0.1286	0.4320	0.7388	0.9118	0.9771	0.9953

Table 4.7: Minimum ratio of true δ to required δ_0 for the acceptability of a lot with producers risk of 0.05 for $\alpha = 2$ and $\beta = 2$

p^*	c	$\frac{t}{\delta_0}$							
		0.9	1.05	1.4	1.65	1.9	2.15	2.4	2.65
0.75	0	2.03	2.26	2.85	3.25	3.74	4.23	4.72	4.7
	1	1.74	1.93	2.37	2.69	2.97	3.13	3.5	3.87
	2	1.65	1.81	2.18	2.46	2.7	2.95	3.29	3.37
	3	1.58	1.73	2.06	2.3	2.57	2.8	2.99	3.3
	4	1.54	1.67	2	2.22	2.41	2.64	2.96	3.27
	5	1.52	1.64	1.93	2.16	2.36	2.6	2.8	3.1
	6	1.5	1.62	1.92	2.11	2.33	2.58	2.79	2.99
	7	1.48	1.6	1.9	2.07	2.3	2.49	2.71	2.9
	8	1.47	1.58	1.85	2.05	2.3	2.48	2.72	2.92
	9	1.45	1.56	1.85	2.03	2.24	2.44	2.65	2.85
10	1.44	1.56	1.81	2.02	2.23	2.42	2.67	2.87	
0.9	0	2.14	2.39	3.04	3.47	3.87	4.38	4.63	5.11
	1	1.83	2.02	2.48	2.8	3.1	3.5	3.79	4.19
	2	1.54	1.68	1.96	2.21	2.35	2.49	2.78	2.88
	3	1.48	1.58	1.83	2	2.16	2.32	2.42	2.68
	4	1.41	1.52	1.74	1.87	1.99	2.17	2.29	2.4
	5	1.39	1.47	1.68	1.83	1.92	2.09	2.26	2.37
	6	1.35	1.43	1.62	1.72	1.82	1.95	2.04	2.15
	7	1.32	1.41	1.56	1.7	1.82	1.92	2.04	2.15
	8	1.31	1.37	1.52	1.64	1.78	1.86	1.95	2.05
	9	1.24	1.3	1.44	1.5	1.56	1.66	1.74	1.83
10	1.23	1.29	1.39	1.48	1.56	1.63	1.76	1.83	
0.95	0	2.07	2.31	2.9	3.19	3.68	4.03	4.5	4.97
	1	1.72	1.89	2.3	2.53	2.8	3.07	3.36	3.55
	2	1.59	1.72	2.03	2.28	2.42	2.64	2.94	3.07
	3	1.52	1.63	1.91	2.05	2.2	2.41	2.58	2.83
	4	1.46	1.56	1.8	1.96	2.07	2.29	2.42	2.53
	5	1.41	1.5	1.74	1.87	1.99	2.17	2.33	2.49
	6	1.38	1.47	1.67	1.81	1.91	2.09	2.18	2.2
	7	1.36	1.44	1.62	1.76	1.85	2	2.14	2.25
	8	1.34	1.41	1.59	1.7	1.81	1.92	2.1	2.15
	9	1.33	1.39	1.56	1.67	1.76	1.89	2.01	2.11
10	1.31	1.38	1.53	1.65	1.73	1.83	1.98	2.03	
0.99	0	2.11	2.37	3.01	3.41	3.84	4.34	4.64	5.13
	1	1.8	1.97	2.38	2.68	2.98	3.3	3.54	3.91
	2	1.67	1.81	2.16	2.39	2.67	2.89	3.05	3.21
	3	1.56	1.7	2	2.19	2.42	2.58	2.78	2.98
	4	1.52	1.62	1.87	2.1	2.25	2.44	2.62	2.83
	5	1.48	1.58	1.83	2	2.16	2.3	2.52	2.58
	6	1.44	1.54	1.78	1.91	2.07	2.21	2.34	2.49
	7	1.41	1.51	1.72	1.87	1.99	2.13	2.3	2.45
	8	1.39	1.48	1.69	1.8	1.92	2.09	2.22	2.33
	9	1.36	1.46	1.64	1.76	1.88	2.01	2.11	2.25
10	1.35	1.44	1.62	1.73	1.85	1.95	2.1	2.18	

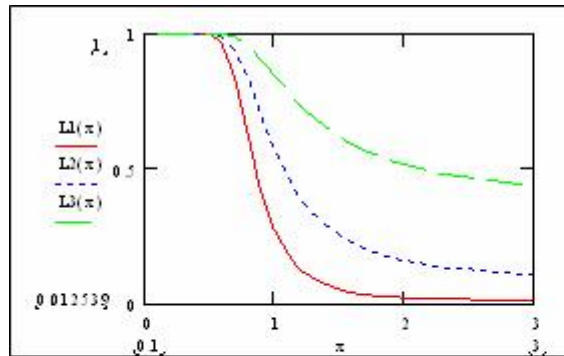


Figure 4.1: OC curve for different samples and for $p^* = 0.75$ and $c=2$

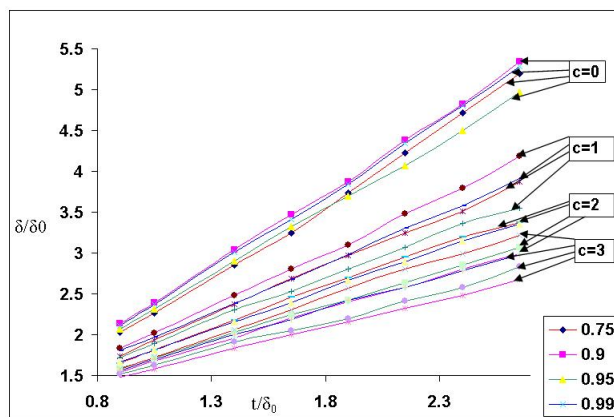


Figure 4.2: minimum ratio of δ/δ_0 with producers risk of 0.05 for $c=1,2$ and 3.

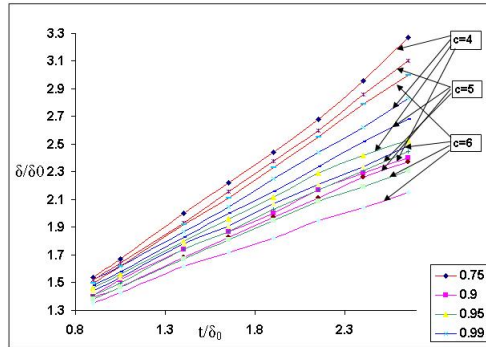


Figure 4.3: minimum ratio of δ/δ_0 with producers risk of 0.05 for $c=4,5$ and 6.

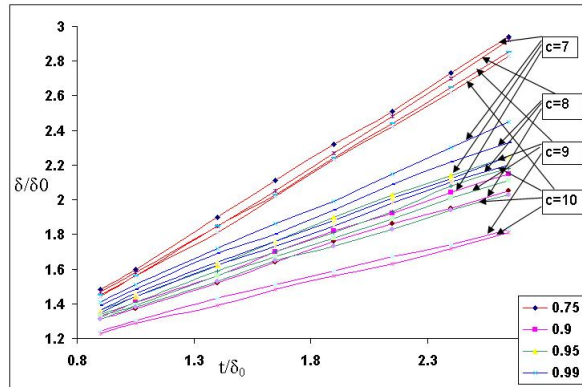


Figure 4.4: minimum ratio of δ/δ_0 with producers risk of 0.05 for $c=7$ to 10.

4.4 Application in time series modeling

Now we discuss various Auto Regressive models of order 1 with Marshall-Olkin Extended Fréchet distribution as marginals, namely MIN AR(1) model I and II and MAX-MIN AR(1) model I and II and explore some properties.

4.4.1 MIN AR(1) model-I with Marshall-Olkin Extended Fréchet marginal distribution

Consider an AR(1) structure given by

$$X_n = \begin{cases} \varepsilon_n & \text{with probability } p \\ \min(X_{n-1}, \varepsilon_n) & \text{with probability } 1 - p \end{cases} \quad (4.4.1)$$

where $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables independent of $\{X_n\}$ and $p \in (0, 1)$. Then the process is stationary Markovian with Marshall-Olkin distribution as marginal.

Theorem 4.4.1. *In an AR(1) process with structure (4.4.1), $\{X_n\}$ is stationary Markovian with Marshall-Olkin Fréchet distribution with parameters p , δ and β if and only if $\{\varepsilon_n\}$ is distributed as Fréchet distribution with parameters δ and β .*

Proof: Sufficiency: Let ε_n follows Fréchet distribution with parameters δ and β . From (4.4.1) it follows that

$$\overline{F}_{X_n}(x) = p\overline{F}_{\varepsilon_n}(x) + (1 - p)\overline{F}_{X_{n-1}}(x)\overline{F}_{\varepsilon_n}(x). \quad (4.4.2)$$

Under stationarity equilibrium, this gives

$$\overline{F}_X(x) = \frac{p\overline{F}_\varepsilon(x)}{1 - (1 - p)\overline{F}_\varepsilon(x)},$$

which is of the Marshall-Olkin form. Necessary: Let X_n follows Marshall-Olkin Extended

Fréchet distribution with parameters p , δ and β . From (4.4.2) under stationarity, we have

$$\bar{F}_\varepsilon(x) = \frac{\bar{F}_X(x)}{p + (1-p)\bar{F}_X(x)}.$$

On simplification we get $\bar{F}_\varepsilon(x) = 1 - e^{-(\frac{\delta}{x})^\beta}$, which is the survival function of Fréchet distribution with parameters δ and β . Let us first consider the joint survival function of random variables X_{n+k} and X_n , $k \geq 1$. We have

$$\begin{aligned} S_k(x, y) &\equiv P(X_{n+k} > x, X_n > y) \\ &= p\bar{F}_\varepsilon(x)\bar{F}_X(y) + (1-p)\bar{F}_\varepsilon(x)S_{k-1}(x, y) \\ &= p\bar{F}_\varepsilon(x)\bar{F}_X(y) \sum_{j=0}^{k-1} (1-p)^j \bar{F}_\varepsilon^j(x) + (1-p)^k \bar{F}_\varepsilon^k(x) S_0(x, y) \\ &= p\bar{F}_\varepsilon(x)\bar{F}_X(y) \frac{1 - (1-p)^k \bar{F}_\varepsilon^k(x)}{1 - (1-p)\bar{F}_\varepsilon(x)} + (1-p)^k \bar{F}_\varepsilon^k(x) S_0(x, y), \end{aligned}$$

where

$$S_0(x, y) = P(X_n > \max(x, y)) = \begin{cases} \bar{F}_X(x), & x \geq y, \\ \bar{F}_X(y), & x < y. \end{cases}$$

Letting $k \rightarrow \infty$, we obtain

$$S_\infty(x, y) = \frac{p\bar{F}_\varepsilon(x)\bar{F}_X(y)}{1 - (1-p)\bar{F}_\varepsilon(x)},$$

i.e. we can see the joint survival function of random variables X_{n+k} and X_n can be represented as a product of two survival function of random variables with parameters p , δ and β . Now we will show that the joint survival function of random variables X_{n+k} and X_n is not a continuous function, i.e. we will show that the probability $P(X_{n+k} = X_n)$ is positive.

We have

$$\begin{aligned}
 P(X_{n+k} = X_n) &= (1-p)P(X_{n+k-1} = X_n, X_{n+k-1} < \varepsilon_{n+k}) \\
 &= (1-p)^2 P(X_{n+k-2} = X_n, X_{n+k-2} < \varepsilon_{n+k-1}, X_{n+k-2} < \varepsilon_{n+k}) \\
 &= (1-p)^k P(X_n < \min(\varepsilon_{n+1}, \dots, \varepsilon_{n+k-1}, \varepsilon_{n+k})). \tag{4.4.3}
 \end{aligned}$$

Now, since random variables ε_{n+i} , $i = 1, 2, \dots, k$, have the survival function $\overline{F}_\varepsilon(x)$, it follows that a random variable $\min(\varepsilon_{n+1}, \dots, \varepsilon_{n+k-1})$ has the survival function $\overline{F}_\varepsilon^k(x)$. Using this, we obtain

$$\begin{aligned}
 P(X_n < \min(\varepsilon_{n+i}, i = 1, 2, \dots, k)) &= \int_0^\infty \overline{F}_\varepsilon^k(x) f_X(x) dx \\
 &= p \int_0^\infty \overline{F}_\varepsilon^k(x) \frac{f_\varepsilon(x)}{(1 - (1-p)\overline{F}_\varepsilon(x))^2} dx \\
 &= 1 - \frac{pk}{k+1} {}_2F_1(1, 1+k; 2+k; 1-p) \tag{4.4.4}
 \end{aligned}$$

Finally, replacing (4.4.4) in (4.4.3), we obtain that the probability $P(X_{n+k} = X_n)$ is positive. Now we will derive the probability of the event $\{X_{n+k} > X_n\}$, $k \geq 1$. We have

$$\begin{aligned}
 P(X_{n+k} > X_n) &= pP(\varepsilon_{n+k} > X_n) + (1-p)P(\min(X_{n+k-1}, \varepsilon_{n+k}) > X_n) \\
 &= p \sum_{j=0}^{k-1} (1-p)^j P(\min(\varepsilon_{n+k-j}, \dots, \varepsilon_{n+k}) > X_n),
 \end{aligned}$$

since the probability of the event $\{\min(X_n, \varepsilon_{n+k-j}, \dots, \varepsilon_{n+k}) > X_n\}$ is 0. Using (4.4.4) we obtain

$$P(X_{n+k} > X_n) = p \sum_{j=0}^{k-1} (1-p)^j \left(1 - \frac{p(j+1)}{j+2} {}_2F_1(1, j+2; j+3; 1-p) \right).$$

For $k = 1$ we have

$$P(X_{n+1} > X_n) = p \left(1 - \frac{p}{2} {}_2F_1(1, 2; 3; 1-p) \right) = \frac{p(1-p+p \log p)}{(1-p)^2}.$$

This probability is an increasing function on p . Also, we can see that it takes values from $(0, \frac{1}{2})$. Thus we can conclude that as p increases that we can observe more down runs of the process $\{X_n\}$.

4.4.2 MIN AR(1) model-II with Marshall-Olkin Fréchet marginal distribution

Now we discuss a more general structure which allows probabilistic selection of process values, innovations and combinations of both. Consider an AR(1) structure given by

$$X_n = \begin{cases} X_{n-1} & \text{with probability } p_1 \\ \varepsilon_n & \text{with probability } p_2 \\ \min(X_{n-1}, \varepsilon_n) & \text{with probability } 1 - p_1 - p_2 \end{cases} \quad (4.4.5)$$

where $p_1, p_2, p_3 > 0$, $p_1 + p_2 < 1$ and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables independent of $\{X_n\}$. Then the process is stationary Markovian with Marshall-Olkin distribution as marginal.

Theorem 4.4.2. *In an AR(1) process with structure (4.4.5), $\{X_n\}$ is stationary Markovian with Marshall-Olkin Extended Fréchet distribution with parameters q, δ, β if and only if $\{\varepsilon_n\}$ is distributed as Fréchet distribution with parameters δ and β , where $q = \frac{p_2}{1-p_1}$.*

Proof: Sufficiency: Let ε_n follows Fréchet distribution with parameters δ and β . From (4.4.5) it follows that

$$\bar{F}_{X_n}(x) = p_1 \bar{F}_{X_{n-1}}(x) + p_2 \bar{F}_{\varepsilon_n}(x) + (1 - p_1 - p_2) \bar{F}_{X_{n-1}}(x) \bar{F}_{\varepsilon_n}(x). \quad (4.4.6)$$

Under stationarity equilibrium, this gives

$$\overline{F}_X(x) = \frac{q\overline{F}_\varepsilon(x)}{1-(1-q)\overline{F}_\varepsilon(x)}, \text{ where } q = \frac{p_2}{1-p_1},$$

which is of the Marshall–Olkin form.

Necessary: Let X_n follows Marshall-Olkin Fréchet distribution with parameters q , δ and β .

From (4.4.6) under stationarity, we have that

$$\overline{F}_\varepsilon(x) = \frac{(1-p_1)\overline{F}_X(x)}{p_2 + (1-p_1-p_2)\overline{F}_X(x)}.$$

Now, using the fact that X_n has the Marshall-Olkin Extended Fréchet distribution with parameters q , δ and β , we get that

$$\overline{F}_\varepsilon(x) = 1 - e^{-\left(\frac{\delta}{x}\right)^\beta},$$

which is the survival function of Fréchet distribution with parameters δ and β . Let us first consider the joint survival function of random variables X_{n+k} and X_n , $k \geq 1$. We have

$$\begin{aligned} S_k(x, y) &= p_2\overline{F}_\varepsilon(x)\overline{F}_X(y) + [p_1 + (1-p_1-p_2)\overline{F}_\varepsilon(x)]S_{k-1}(x, y) \\ &= p_2\overline{F}_\varepsilon(x)\overline{F}_X(y) \sum_{j=0}^{k-1} [p_1 + (1-p_1-p_2)\overline{F}_\varepsilon(x)]^j \\ &\quad + [p_1 + (1-p_1-p_2)\overline{F}_\varepsilon(x)]^k S_0(x, y) \\ &= p_2\overline{F}_\varepsilon(x)\overline{F}_X(y) \frac{1 - [p_1 + (1-p_1-p_2)\overline{F}_\varepsilon(x)]^k}{1 - p_1 - (1-p_1-p_2)\overline{F}_\varepsilon(x)} \\ &\quad + [p_1 + (1-p_1-p_2)\overline{F}_\varepsilon(x)]^k S_0(x, y). \end{aligned}$$

As in the case when $p_1 = 0$, letting $k \rightarrow \infty$, we obtain that the joint survival function of random variables X_{n+k} and X_n can be represented as a product of two survival function of random variables with parameters q , δ and β . Let us consider now the probability $P(X_{n+k} = X_n)$. To simplify the derivations, we will denote by A_{i_1, \dots, i_r}^j the event

$\{X_{n+j} = X_n, X_{n+j} < \min(\varepsilon_{n+i_1}, \dots, \varepsilon_{n+i_r})\}$. We have

$$\begin{aligned}
 P(X_{n+k} = X_n) &= p_1 P(X_{n+k-1} = X_n) + (1 - p_1 - p_2) P(A_k^{k-1}) \\
 &= p_1^2 P(X_{n+k-2} = X_n) + p_1(1 - p_1 - p_2) P(A_{k-1}^{k-2}) \\
 &\quad + p_1(1 - p_1 - p_2) P(A_k^{k-2}) + (1 - p_1 - p_2)^2 P(A_{k-1,k}^{k-2}) \\
 &= p_1^k + p_1^{k-1}(1 - p_1 - p_2) \sum_{i_1=1}^k P(A_{i_1}^0) \\
 &\quad + p_1^{k-2}(1 - p_1 - p_2)^2 \sum_{i_1 < i_2} P(A_{i_1, i_2}^0) + \dots \\
 &\quad + p_1(1 - p_1 - p_2)^{k-1} \sum_{i_1 < \dots < i_{k-1}} P(A_{i_1, \dots, i_{k-1}}^0) \\
 &\quad + (1 - p_1 - p_2)^k P(A_{i_1, \dots, i_k}^0). \tag{4.4.7}
 \end{aligned}$$

From (4.4.4) we have

$$P(A_{i_1, \dots, i_r}^0) = 1 - \frac{qr}{1+r} {}_2F_1(1, 1+r; 2+r; 1-q).$$

Replacing this in (4.4.7), we obtain that the probability of the event $\{X_{n+k} = X_n\}$ is

$$P(X_{n+k} = X_n) = \sum_{j=0}^k p_1^j (1 - p_1 - p_2)^{k-j} \binom{k}{j} \left[1 - \frac{qj}{1+j} {}_2F_1(1, 1+j; 2+j; 1-q) \right].$$

Now we will derive the probability of the event $\{X_{n+1} > X_n\}$. From the definition of the process and (4.4.4), we have

$$P(X_{n+1} > X_n) = p_2 P(\varepsilon_{n+1} > X_n) = \frac{p_2(1 - q + q \log q)}{(1 - q)^2}.$$

4.4.3 MAX-MIN AR(1) model-I with Marshall-Olkin Extended Fréchet marginal distribution

Consider the AR(1) structure given by

$$X_n = \begin{cases} \max(X_{n-1}, \varepsilon_n) & \text{with probability } p_1 \\ \min(X_{n-1}, \varepsilon_n) & \text{with probability } p_2 \\ X_{n-1} & \text{with probability } 1 - p_1 - p_2, \end{cases} \quad (4.4.8)$$

where $0 < p_1, p_2 < 1$, $p_2 < p_1$, $p_1 + p_2 < 1$ and $\{\varepsilon_n\}$ is a sequence of i.i.d. random variables independently distributed of X_n . Then the process is stationary Markovian with Marshall-Olkin distribution as marginal.

Theorem 4.4.3. *In AR(1) Max-Min process with structure (4.4.8), $\{X_n\}$ is a stationary Markovian AR(1) Max-Min process with Marshall-Olkin Fréchet distribution with parameters q , δ and β if and only if $\{\varepsilon_n\}$ follows Fréchet distribution with parameters δ and β , where $q = \frac{p_1}{p_2}$.*

Proof: Sufficiency: Let ε_n follows Fréchet distribution with parameters δ and β . From structure (4.4.8), we have

$$\begin{aligned} \overline{F}_{X_n}(x) = & p_1[1 - (1 - \overline{F}_{X_{n-1}}(x))(1 - \overline{F}_{\varepsilon_n}(x))] + p_2\overline{F}_{X_{n-1}}(x)\overline{F}_{\varepsilon_n}(x) \\ & + (1 - p_1 - p_2)\overline{F}_{X_{n-1}}(x). \end{aligned} \quad (4.4.9)$$

Under stationary equilibrium,

$$\overline{F}_{X_n}(x) = \frac{q\overline{F}_{\varepsilon}(x)}{1 - (1 - q)\overline{F}_{\varepsilon}(x)},$$

where $q = p_1/p_2$ and it is in the Marshall-Olkin form.

Necessary: Let X_n follows Marshall-Olkin Extended Fréchet distribution with parameters

q , δ and β . Then from (4.4.9) under stationarity

$$\bar{F}_\varepsilon(x) = \frac{p_2 \bar{F}_{X_n}(x)}{p_1 + (p_2 - p_1) \bar{F}_{X_n}(x)}.$$

On simplification we get

$$\bar{F}_\varepsilon(x) = 1 - e^{-(\frac{\delta}{x})^\beta}$$

which is the survival function of Fréchet distribution with parameters δ and β . In many situations of practical interest is the probability of the event $\{X_{n+1} > X_n\}$. After some calculations, we can show that

$$P(X_{n+1} > X_n) = p_1 P(\varepsilon_{n+1} > X_n) = \frac{p_1(1 - q + q \log q)}{(1 - q)^2},$$

4.4.4 MAX-MIN AR(1) model-II with Marshall-Olkin Extended Fréchet marginal distribution

Finally we consider more general Max-Min process which includes maximum, minimum as well as the innovations and the process. The AR(1) structure is given by

$$X_n = \begin{cases} \max(X_{n-1}, \varepsilon_n) & \text{with probability } p_1 \\ \min(X_{n-1}, \varepsilon_n) & \text{with probability } p_2 \\ \varepsilon_n & \text{with probability } p_3 \\ X_{n-1} & \text{with probability } 1 - p_1 - p_2 - p_3, \end{cases} \quad (4.4.10)$$

where $0 < p_1, p_2, p_3 < 1$, $p_1 + p_2 + p_3 < 1$ and $\{\varepsilon_n\}$ is a sequence of i.i.d. random variables independently distributed of X_n . Then the process is stationary Markovian with Marshall-Olkin distribution as marginal.

Theorem 4.4.4. *AR(1) Max-Min process $\{X_n\}$ with structure (4.4.10) is a station-*

ary Markovian AR(1) Max-Min process with Marshall-Olkin Fréchet distribution with parameters q , δ and β if and only if $\{\varepsilon_n\}$ follows Fréchet distribution with parameters δ and β , where $q = \frac{p_1+p_3}{p_2+p_3}$.

Proof: Sufficiency: Let ε_n follows Fréchet distribution with parameters δ and β . From structure (4.4.10), we have

$$\begin{aligned} \bar{F}_{X_n}(x) = & p_1[1 - (1 - \bar{F}_{X_{n-1}}(x))(1 - \bar{F}_{\varepsilon_n}(x))] + p_2\bar{F}_{X_{n-1}}(x)\bar{F}_{\varepsilon_n}(x) \\ & + p_3\bar{F}_{\varepsilon_n}(x) + (1 - p_1 - p_2 - p_3)\bar{F}_{X_{n-1}}(x). \end{aligned} \quad (4.4.11)$$

Under stationary equilibrium, we obtain

$$\bar{F}_{X_n}(x) = \frac{q\bar{F}_{\varepsilon}(x)}{1 - (1 - q)\bar{F}_{\varepsilon}(x)}, \quad q = \frac{p_1 + p_3}{p_2 + p_3}$$

which is of the Marshall-Olkin form.

Necessary: Let X_n follows Marshall-Olkin ExtendedFréchet distribution with parameters q , δ and β . Then from(4.4.11) under stationarity, we have

$$\bar{F}_{\varepsilon}(x) = \frac{(p_2 + p_3)\bar{F}_X(x)}{(p_1 + p_3) + (p_2 - p_1)\bar{F}_X(x)}.$$

On simplification, we get

$$\bar{F}_{\varepsilon}(x) = 1 - e^{-\left(\frac{\delta}{x}\right)^{\beta}},$$

which is the survival function of Fréchet distribution with parameters δ and β .

Remark

The above model can describe the response to treatment of a patient suffering from B.P. In normal situation X_n is same as X_{n-1} . For an acute patient always the innovation ε_n is important. In some cases we have to keep the minimum as well as maximum at a particular level.

4.4.5 Sample path behavior

Figure 4.5 - 4.8 describe the sample path properties of the four AR(1) models developed above

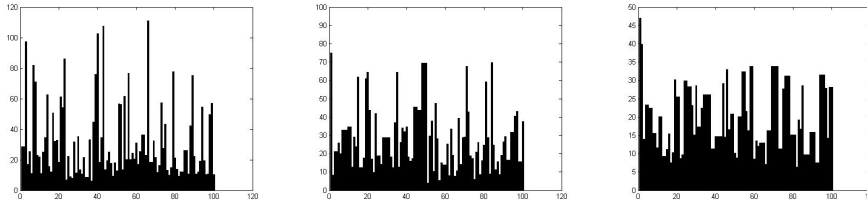


Figure 4.5: Sample path for AR(1) Minification model I for various values of $p = 0.9, 0.8, 0.6$ and $\delta = 20$ and $\beta = 1.2$

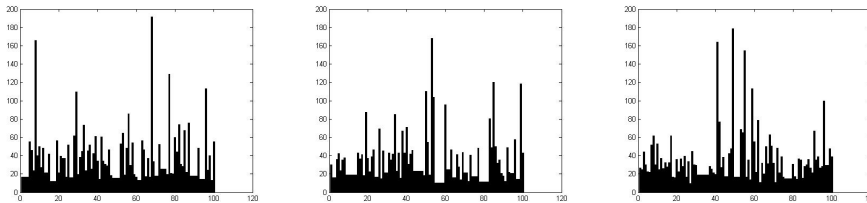


Figure 4.6: Sample path for AR(1) Minification model II for various combinations of $(p_1, p_2) = (0.4, 0.2), (0.2, 0.4), (0.2, 0.2)$ and $\delta = 25$ and $\beta = 2$

Conclusion: The continuous improvement and review of acceptance sampling plan is important to improve the quality of the products and to ensure customer satisfaction.

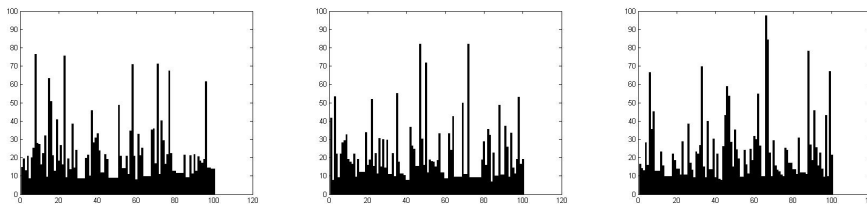


Figure 4.7: Sample path for AR(1) Minmax model I for various combinations of $(p_1, p_2) = (0.2, 0.3), (0.3, 0.2), (0.3, 0.3)$ and $\delta = 15$ and $\beta = 2$

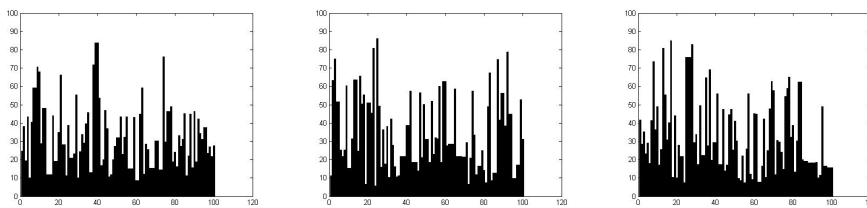


Figure 4.8: Sample path for AR(1) Minmax model II for various combinations of $(p_1, p_2, p_3) = (0.5, 0.2, 0.1), (0.3, 0.2, 0.3), (0.4, 0.2, 0.2)$ and $\delta = 25$ and $\beta = 1.2$

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CHAPTER 5

Marshall-Olkin Exponentiated Generalized Exponential Distribution and its Applications

5.1 Introduction

The exponential distribution is one of the most significant and widely used distributions in statistical literature and it possesses several important statistical properties. It is the most commonly used distribution in reliability studies, even though its hazard rate function is constant. The two-parameter generalized exponential distribution has been studied extensively by Gupta and Kundu (1999, 2001, 2002, 2003, 2004), Raqab (2002), Raqab and Ahsanullah (2001). The generalized exponential distribution is a sub-model of the exponentiated Weibull distribution introduced by Mudholkar and Srivastava (1993) and later

studied by Mudholkar, Srivastava and Freimer (1995) and Mudholkar and Hutson (1996). The generalized exponential distribution is also known as exponentiated exponential and it has two parameters (scale and shape) similar to the Weibull or gamma family. Alice and Jose(2005) discusses Marshall-Olkin semi-Weibull minification processes and Jose et al.(2010) also proposed Marshall-Olkin q-Weibull distribution and maxmin processes. Jayakumar and Thomas(2008) developed a generalization of Marshall-Olkin distribution and discussed its application to Burr Type XII distribution. Jose et al.(2011) discussed Marshall-Olkin bivariate Weibull distributions and applications. Barriga, Louzada and Cancho (2011) deal with the Complementary Exponential Power Lifetime model which is obtained by exponentiation of the Exponential Power distribution proposed by Smith and Bain (1975). Krishna et al.(2013 a,b) introduced Marshall-Olkin Fréchet distribution and discussed its applications. Bakouch et al.(2012) deal with the extension of the Lindley distribution. Nadarajah (2005) discussed the exponentiated Gumbel distribution. Shirke and Kakade (2006) dicussed exponentiated log-normal and Nadarajah and Gupta (2007) the exponentiated gamma distributions.

In this chapter we consider Marshall-Olkin Exponentiated Generalized Exponential Distribution and its Applications. Exponentiated generalized Exponential distribution and its properties are considered in detail. A new distribution called Marshall-Olkin Exponentiated Generalized Exponential Distribution is introduced and its properties are discussed. The quantiles and order statistics are considered. The maximum likelihood estimates are obtained and applied to a real data set on carbon fibers. Reliability of a system following Marshall-Olkin Exponentiated generalized Exponential distribution under stress-strength model is estimated and its validity is measured in terms of average bias and average mean square error calculated from the simulated N estimates. The average length of the 95% asymptotic confidence intervals and coverage probability for the estimates obtained by simulation are evaluated.

5.2 Exponentiated Generalized Exponential Distribution

The cumulative distribution function(c.d.f.) of the exponential distribution is $G(x) = 1 - \exp(-\lambda x)$ where $\lambda > 0$. Then define the Exponentiated Generalized Exponential (EGE) with cumulative distribution given by

$$F(x) = [1 - \exp(-\alpha\lambda x)]^\beta \quad (5.2.1)$$

where $x > 0, \alpha > 0, \beta > 0, \lambda > 0$ and the corresponding density function can be obtained from (1.3.4) as

$$f(x) = \alpha\beta\lambda \exp(-\lambda\alpha x)[1 - \exp(-\alpha\lambda x)]^{\beta-1}$$

where $x > 0, \alpha > 0, \beta > 0, \lambda > 0$. It is denoted as $EGE(\alpha, \beta, \lambda)$.

The survival function and Hazard rate function of the EGE distribution is given by

$$\bar{F}(x) = 1 - [1 - \exp(-\alpha\lambda x)]^\beta \quad (5.2.2)$$

$$r(x) = \frac{\alpha\beta\lambda \exp(-\alpha\lambda x)[1 - \exp(-\alpha\lambda x)]^{\beta-1}}{1 - [1 - \exp(-\alpha\lambda x)]^\beta}$$

Figure 5.1 and Figure 5.2 shows the pdf and hazard function of EGE distribution for different values of parameters.

The p^{th} quantile function x_p of the EGE distribution, the inverse of the distribution function $F(x_p) = p$ is given by

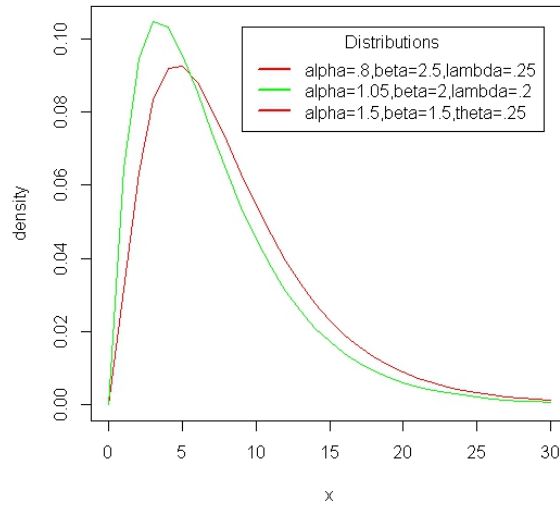


Figure 5.1: The p.d.f. of EGE distribution for different values of parameters

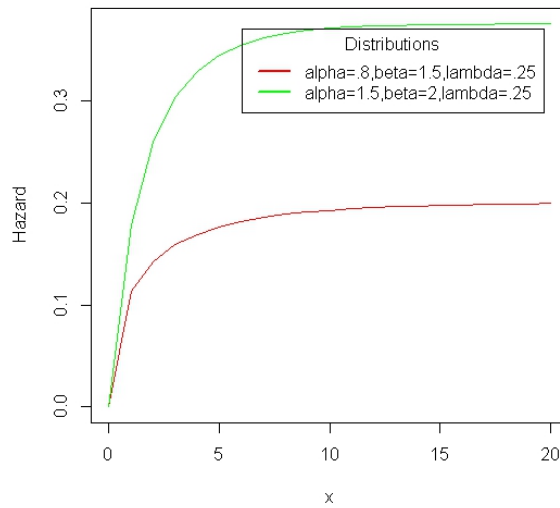


Figure 5.2: Hazard rate function of EGE distribution for different values of parameters

$$x_p = \frac{-1}{\alpha\lambda} \log(1 - p^{\frac{1}{\beta}})$$

5.3 Marshall-Olkin Exponentiated Generalized Exponential Distribution

Consider the survival function of exponentiated generalized exponential distribution $\bar{F}(x) = 1 - [1 - \exp(-\alpha\lambda x)]^\beta$. In this section using Marshall-Olkin techniques, we introduce Marshall-Olkin Exponentiated Generalized Exponential (MOEGE) Distribution. The survival function of MOEGE distribution is given by

$$\bar{G}(x) = \frac{\theta - \theta[1 - \exp(-\alpha\lambda x)]^\beta}{\theta + (1 - \theta)[1 - \exp(-\alpha\lambda x)]^\beta} \quad (5.3.1)$$

Then the corresponding probability density function is given by

$$g(x) = \frac{\theta\alpha\beta\lambda \exp(-\alpha\lambda x)[1 - \exp(-\alpha\lambda x)]^{\beta-1}}{[1 - (1 - \theta)(1 - [1 - \exp(-\alpha\lambda x)]^\beta)]^2} \quad (5.3.2)$$

$\alpha > 0, \beta > 0, \lambda > 0, \theta > 0$. It is denoted as $MOEGE(\alpha, \beta, \lambda, \theta)$.

1. When $\theta = 1$ the above density becomes exponentiated generalized exponential, and when $\theta = \alpha = 1$ it becomes Generalized Exponential distribution.
2. When $\theta = \alpha = \beta = 1$ the distribution becomes exponential distribution and when $\alpha = \beta = 1$ it becomes Marshall-Olkin Exponential distribution.

The cumulative distribution function is given by

$$G(x) = \frac{[1 - \exp(-\alpha\lambda x)]^\beta}{\theta + (1 - \theta)[1 - \exp(-\alpha\lambda x)]^\beta}$$

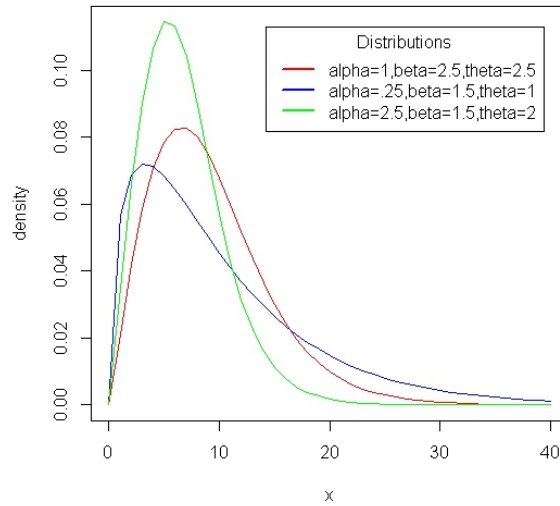


Figure 5.3: The p.d.f. of MOEGE distribution for different values of parameters

and the hazard rate function is given by

$$h(x) = \frac{\alpha\beta\lambda \exp(-\alpha\lambda x)[1 - \exp(-\alpha\lambda x)]^{\beta-1}}{[1 - (1 - \exp(-\alpha\lambda x))^{\beta}][1 - (1 - \theta)[1 - (1 - \exp(-\alpha\lambda x))^{\beta}]]}$$

Figure 5.3 and Figure 5.4 shows the pdf and hazard function of different combinations of parameter values.

5.4 Quantiles and Order statistics

The p^{th} quantile function of the distribution, the inverse of the distribution function $F(x_p) = p$, is given by

$$x_p = \frac{-1}{\alpha\lambda} \log \left(1 - \left[\frac{p\theta}{1 - p(1 - \theta)} \right]^{1/\beta} \right)$$

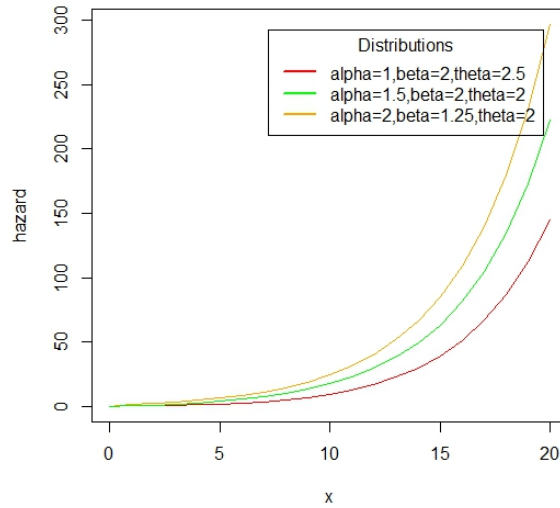


Figure 5.4: Hazard rate function of MOEGE distribution for different values of parameters

Let X_1, X_2, \dots, X_n be a random sample taken from the MOEGE distribution and $X_{1:n}, X_{2:n}, \dots, X_{n:n}$ be the corresponding order statistics. The survival function of MOEGE distribution is given by (5.3.1). Then the c.d.f. of the first order statistic $X_{1:n}$ is given by

$$\begin{aligned} G_{1:n}(x) &= 1 - (\bar{G}(x))^n \\ &= 1 - \left[\frac{\theta - \theta[1 - \exp(-\alpha\lambda x)]^\beta}{\theta + (1 - \theta)[1 - \exp(-\alpha\lambda x)]^\beta} \right]^n \end{aligned}$$

The c.d.f. of the n^{th} order statistic $X_{n:n}$ is given by

$$\begin{aligned} G_{n:n}(x) &= [1 - \bar{G}(x)]^n \\ &= \frac{[1 - \exp(-\alpha\lambda x)]^{n\beta}}{[\theta + (1 - \theta)[1 - \exp(-\alpha\lambda x)]^\beta]^n} \end{aligned}$$

The probability density function $g_{i:n}(x)$ of the i^{th} order statistics $X_{i:n}$ is given by

$$g_{i:n}(x) = \frac{n!}{(i-1)!(n-1)!} \frac{\theta\alpha\beta\lambda \exp(-\alpha\lambda x)[1 - \exp(-\alpha\lambda x)]^{i\beta-1}}{[1 - (1-\theta)(1 - [1 - \exp(-\alpha\lambda x)]^\beta)]^2} \frac{[\theta - \theta[1 - \exp(-\alpha\lambda x)]^\beta]^{n-i}}{[\theta + (1-\theta)[1 - \exp(-\alpha\lambda x)]^\beta]^{n-1}}$$

5.5 Estimation of Parameters

In this section we consider maximum likelihood estimation for a given random sample (x_1, x_2, \dots, x_n) . Then the log likelihood function is given by

$$\begin{aligned} \log L(\alpha, \beta, \lambda, \theta) &= n(\log\theta + \log\alpha + \log\beta + \log\lambda) + \alpha\lambda \sum_{i=1}^n x_i + \\ &(\beta - 1) \sum_{i=1}^n \log(1 - \exp(-\alpha\lambda x_i)) - \\ &2 \sum_{i=1}^n \log [1 - (1 - \theta)(1 - [1 - \exp(-\alpha\lambda x_i)]^\beta)] \end{aligned}$$

The partial derivative of the log likelihood functions with respect to the parameters are

$$\begin{aligned} \frac{\partial \log L}{\partial \alpha} &= \frac{n}{\alpha} + \lambda \sum_{i=1}^n x_i + \lambda(\beta - 1) \sum_{i=1}^n \frac{x_i \exp(-\alpha\lambda x_i)}{1 - \exp(-\alpha\lambda x_i)} - \\ &2(1 - \theta)\beta\lambda \sum_{i=1}^n \frac{x_i \exp(-\alpha\lambda x_i)[1 - \exp(-\alpha\lambda x_i)]^{\beta-1}}{1 - (1 - \theta)[1 - [1 - \exp(-\alpha\lambda x_i)]^\beta]} \end{aligned}$$

$$\begin{aligned} \frac{\partial \log L}{\partial \beta} &= \frac{n}{\beta} + \sum_{i=1}^n \log[1 - \exp(-\alpha\lambda x_i)] - \\ &2(1 - \theta) \sum_{i=1}^n \frac{[1 - \exp(-\alpha\lambda x_i)]^\beta \log[1 - \exp(-\alpha\lambda x_i)]}{1 - (1 - \theta)[1 - [1 - \exp(-\alpha\lambda x_i)]^\beta]} \end{aligned}$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} + \alpha \sum_{i=1}^n x_i + (\beta - 1)\alpha \sum_{i=1}^n \frac{x_i \exp(-\alpha \lambda x_i)}{1 - \exp(-\alpha \lambda x_i)} - 2\alpha\beta(1 - \theta) \sum_{i=1}^n \frac{x_i \exp(-\alpha \lambda x_i) [1 - \exp(-\alpha \lambda x_i)]^{\beta-1}}{1 - (1 - \theta) [1 - [1 - \exp(-\alpha \lambda x_i)]^\beta]}$$

$$\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} - 2 \sum_{i=1}^n \frac{(1 - [1 - \exp(-\alpha \lambda x_i)]^\beta)}{1 - (1 - \theta) [1 - [1 - \exp(-\alpha \lambda x_i)]^\beta]}$$

The maximum likelihood estimates can be obtained by solving the normal equations $\frac{\partial \log L}{\partial \alpha} = 0, \frac{\partial \log L}{\partial \beta} = 0, \frac{\partial \log L}{\partial \lambda} = 0, \frac{\partial \log L}{\partial \theta} = 0$. The equations can be solved using nlm package in R software.

5.5.1 Data Analysis

In this section we present the analysis of one real data set. It is a strength data considered by Badar and Priest(2011). The data represent the strength measured in GPA, for single carbon fibers. Single fibers were tested under tension at gauge lengths of 1, 10, 20 and 50 mm. We are considering the single fibers data set of 10 mm in gauge lengths with sample size 63. The data are presented below.

1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.

Exponentiated Generalized Exponential with α, β and λ , Exponential with λ and Marshall-Olkin Exponentiated Generalized Exponential distribution with parameters α, β, λ and θ are fitted to the data. Estimates of the parameters are shown in Table 5.1. From

Table 5.1: Summary of fitting for the MOEGE, EGE and Exponential distribution.

Model	Parameter	Estimates	-Log-likelihood	K-S statistic
MOEGE	α	0.7780	74.9965	0.4605
	β	0.7975		
	λ	0.1670		
	θ	0.7895		
EGE	α	0.9180	120.0223	0.8253
	β	0.8850		
	λ	0.0119		
Exponential	λ	0.3742	77.3343	0.5240

Table 5.1, it is seen that the K-S statistic for the MOEGE distribution is 0.4605 which is less than that for the EGE and Exponential distribution. Similar is the case with log likelihood values. Hence we conclude that MOEGE distribution is a better model for the data set.

5.6 Stress-Strength Analysis

The stress-strength reliability analysis can be described as an assessment of reliability of a system in terms of random variables X and Y , where X represents stress and Y represents the strength. If the stress exceeds strength the system would fail and the system will function if strength exceeds stress. The stress-strength reliability can be defined as $R = P(X < Y)$. Gupta et al.(2010) obtained various results on the MO family in the context of reliability modeling and survival analysis. We have,

$$\begin{aligned}
 P(X < Y) &= \int_{-\infty}^{\infty} P(Y > X/X = x)g_X(x)dx \\
 &= \int_0^{\infty} \frac{\theta - \theta[1 - \exp(-\alpha\lambda x)]^\beta}{\theta + (1 - \theta)[1 - \exp(-\alpha\lambda x)]^\beta} \frac{\theta\alpha\beta \exp(-\alpha\lambda x)[1 - \exp(-\alpha\lambda x)]^{\beta-1}}{[1 - (1 - \theta)(1 - [1 - \exp(-\alpha\lambda x)]^\beta)]^2} dx \\
 &= \frac{\frac{\alpha_1}{\alpha_2}}{(\frac{\alpha_1}{\alpha_2} - 1)^2} \left[-\log \frac{\alpha_1}{\alpha_2} + \frac{\alpha_1}{\alpha_2} - 1 \right]
 \end{aligned}$$

Consider the pdf of Marshall-Olkin Exponentiated Generalized Exponential distribution given by

$$g(x, \alpha, \beta, \lambda, \theta) = \frac{\theta \alpha \beta \exp(-\alpha \lambda x) [1 - \exp(-\alpha \lambda x)]^{\beta-1}}{[1 - (1 - \theta)(1 - [1 - \exp(-\alpha \lambda x)]^\beta)]^2}$$

$\alpha > 0, \beta > 0, \lambda > 0, \theta > 0$.

Let (x_1, x_2, \dots, x_m) and (y_1, y_2, \dots, y_n) be two independent random samples of sizes m and n from Marshall-Olkin Exponentiated Generalized Exponential distribution with tilt parameters α_1 and α_2 respectively, and common unknown parameters β, λ and θ . The log likelihood function is given by

$$L(\alpha_1, \alpha_2, \beta, \lambda, \theta) = \sum_{i=1}^m \log g(x_i; \alpha_1, \beta, \lambda, \theta) + \sum_{j=1}^n \log g(y_j; \alpha_2, \beta, \lambda, \theta)$$

The maximum likelihood estimates of the unknown parameters α_1, α_2 are the solutions of non linear equations $\frac{\partial L}{\partial \alpha_1} = 0$ and $\frac{\partial L}{\partial \alpha_2} = 0$ respectively.

The elements of Information matrix are

$$I_{11} = -E\left(\frac{\partial^2 L}{\partial \alpha_1^2}\right) = \frac{m}{3\alpha_1^2}$$

$$I_{22} = -E\left(\frac{\partial^2 L}{\partial \alpha_2^2}\right) = \frac{n}{3\alpha_2^2}$$

$$I_{12} = I_{21} = -E\left(\frac{\partial^2 L}{\partial \alpha_1 \alpha_2}\right) = 0$$

By the property of m.l.e for $m \rightarrow \infty, n \rightarrow \infty$, we have

$$(\sqrt{m}(\hat{\alpha}_1 - \alpha_1), \sqrt{n}(\hat{\alpha}_2 - \alpha_2)) \xrightarrow{d} N_2(0, \text{diag}\{a_{11}^{-1}, a_{22}^{-1}\})$$

where $a_{11} = \lim_{m,n \rightarrow \infty} \frac{1}{m} I_{11} = \frac{1}{3\alpha_1^2}$ and $a_{22} = \lim_{m,n \rightarrow \infty} \frac{1}{m} I_{22} = \frac{1}{3\alpha_2^2}$.

The 95% confidence interval for R is given by

$$\hat{R} \pm 1.96\hat{\alpha}_1 b_1(\hat{\alpha}_1, \hat{\alpha}_2) \sqrt{\frac{3}{m} + \frac{3}{n}}$$

where $\hat{R} = R(\hat{\alpha}_1, \hat{\alpha}_2)$ is the estimator of R and

$$b_1(\alpha_1, \alpha_2) = \frac{\partial R}{\partial \alpha_1} = \frac{\alpha_2}{(\alpha_1 - \alpha_2)^3} \left[-2(\alpha_1 - \alpha_2) + (\alpha_1 + \alpha_2) \ln \frac{\alpha_1}{\alpha_2} \right]$$

5.6.1 Simulation Study

We generate $N = 10000$ sets of X samples and Y samples from Marshall-Olkin Exponentiated Generalized Exponential distribution with parameters $\alpha_1, \beta, \lambda, \theta$ and $\alpha_2, \beta, \lambda, \theta$ respectively. The combinations of samples of sizes $m = 20, 25, 30$ and $n = 20, 25, 30$ are considered. The estimates of α_1 and α_2 are obtained from each sample to obtain \hat{R} . The validity of the estimate of R is discussed by the measures

1. Average bias of the simulated N estimates of R:

$$\frac{1}{N} \sum_{i=1}^N (\hat{R} - R)$$

2. Average mean square error of the simulated N estimates of R:

$$\frac{1}{N} \sum_{i=1}^N (\hat{R} - R)^2$$

3. Average length of the asymptotic 95% confidence intervals of R:

$$\frac{1}{N} \sum_{i=1}^N 2(1.96)\hat{\alpha}_{1i} b_{1i}(\hat{\alpha}_{1i}, \hat{\alpha}_{2i}) \sqrt{\frac{3}{m} + \frac{3}{n}}$$

4. The coverage probability of the N simulated confidence intervals given by the proportion of such interval that include the parameter R .

The average bias and average mean square error of the simulated estimates of R for various combination values are given in Table 2.2. The average confidence length and coverage probability of the simulated estimates of R for various combination values are given in Table 2.3.

5.7 Conclusion

In this chapter we have introduced Marshall-Olkin Exponentiated Generalized Exponential distribution and studied its properties. We analyze a real data set and compare MOEGE distribution with the exponentiated generalized exponential distribution and exponential distribution. We conclude that the MOEGE distribution is a better fit. The results of the data analysis are summarized in Table 5.1. The stress-strength reliability analysis is also done. In Table 5.2, we get the average bias and average mean square error of the estimates obtained in the simulation study. Also in Table 5.3, we get the average confidence length and coverage probability of the simulated estimates of R for various combinations of parameter values. The MATLAB programme developed for computation is given in section 5.8, as Appendix.

Table 5.2: Average bias and average MSE of the simulated estimates of R for $\beta = 4, \lambda = 3$ and $\theta = 2$

		(α_1, α_2)							
		Average bias (b)				Average Mean Square Error (AMSE)			
(m,n)		(0.8,0.9)	(1.2,1.3)	(1.5,1.9)	(2,1.5)	(0.8,0.9)	(1.2,1.3)	(1.5,1.9)	(2,1.5)
(20,20)		0.0398	0.0265	0.0781	-0.0950	0.0041	0.0033	0.0087	0.0116
(20,25)		0.0391	0.0265	0.0783	-0.0948	0.0041	0.0032	0.0086	0.0114
(20,30)		0.0389	0.0264	0.0781	-0.0940	0.0041	0.0032	0.0086	0.0114
(25,20)		0.0411	0.0281	0.0807	-0.0938	0.0040	0.0031	0.0088	0.0110
(25,25)		0.0416	0.0292	0.0803	-0.0929	0.0040	0.0032	0.0087	0.0109
(25,30)		0.0413	0.0294	0.0809	-0.0921	0.0041	0.0032	0.0088	0.0107
(30,20)		0.0433	0.0302	0.0823	-0.0910	0.0040	0.0030	0.0089	0.0104
(30,25)		0.0434	0.0304	0.0822	-0.0912	0.0040	0.0031	0.0089	0.0104
(30,30)		0.0426	0.0298	0.0821	-0.0913	0.0040	0.0030	0.0089	0.0104

5.8 Appendix

```

s1=0;
s2=0;
c=0;
inter=0;
for i=1:10000;
a1=1.2;
bet=4;
lam=3;
th=2;
m=30;
n=25;
u1=unifrnd(0,1,m,1);
x=(a1.*lam).^(-1).*log(1-(u1.*th./(1-u1+u1.*th)).^(1./bet));
[Aihat]=mle(x,'log(th.*a1.*bet.*lam.*exp(-lam.*a1.*x).*(1-exp(-a1.*lam.*x)).^(bet-1)./(1-(1-th).*(1-(1-exp(-a1.*lam.*x)).^bet)).^2)',0.01);

val=Aihat;
subs=[1;2];
k=accumarray(subs,val);
p1=k(1);
q1=k(2);

```


Table 5.3: Average confidence length and coverage probability of the simulated 95% percentage confidence intervals of R for $\beta = 4, \lambda = 3$ and $\theta = 2$

(α_1, α_2)								
	Average Confidence Length				Coverage Probability			
(m,n)	(0.8,0.9)	(1.2,1.3)	(1.5,1.9)	(2,1.5)	(0.8,0.9)	(1.2,1.3)	(1.5,1.9)	(2,1.5)
(20,20)	0.3540	0.3542	0.3526	0.3512	0.9942	0.9964	0.9645	0.9327
(20,25)	0.3359	0.3362	0.3346	0.3338	0.9903	0.9946	0.9520	0.9135
(20,30)	0.3260	0.3235	0.3220	0.3209	0.9874	0.9942	0.9436	0.8940
(25,20)	0.3361	0.3363	0.3345	0.3341	0.9924	0.9963	0.9508	0.9235
(25,25)	0.3169	0.3171	0.3155	0.3151	0.9901	0.9939	0.9360	0.8996
(25,30)	0.3033	0.3035	0.3020	0.3018	0.9825	0.9909	0.9164	0.8984
(30,20)	0.3235	0.3238	0.3135	0.3220	0.9910	0.9958	0.9397	0.9219
(30,25)	0.3035	0.3037	0.3021	0.3025	0.9872	0.9927	0.9198	0.9121
(30,30)	0.2893	0.2896	0.2880	0.2880	0.9805	0.9894	0.8956	0.9485

```

if (abs(p1-a1)<abs(q1-a1))
    A1hat=p1;
else
    A1hat=q1;
end
%[...] = mle(data, 'logpdf', logpdf, 'logf', logf, 'start', start, ...)
a2=1.3;
u2=unifrnd(0,1,n,1);
y=(a2.*lam).^(-1).*log(1-(u2.*th./(1-u2+u2.*th)).^(1./bet));
[A2hat]=mle(y, 'log(th.*a2.*bet.*lam.*exp(-lam.*a2.*x).*(1-exp(-a2.*lam.*x)).^(bet-1)./(1-(1-th).*(1-(1-exp(-a2.*lam.*x)).^bet)).^2)', 0.01);

val=A2hat;
subs=[1;2];
R=accumarray(subs, val);
p2=R(1);
q2=R(2);
if (abs(p2-a2)<abs(q2-a2))
    A2hat=p2;
else
    A2hat=q2;
end
a=a1/a2;

```

```

r1=(a/(a-1)^2)*(-log(a)+a-1);
A=A1hat/A2hat;
R1=(A/(A-1)^2)*(-log(A)+A-1);
t1=R1-r1;
t2=t1*t1;
s1=s1+t1;
s2=s2+t2;
b1=(A2hat/(A1hat-A2hat)^3)*(-2*(A1hat-A2hat)+ (A1hat-A2hat)*log(A1hat/A2hat));
inter=inter+2*1.96*A1hat*b1*sqrt((3/m)+(3/n));
if((r1>R1-1.96*A1hat*b1*sqrt((3/m)+(3/n))&& (r1>R1+1.96*A1hat*b1*sqrt((3/m)+(3/n)))));
c=c+1;
end;
end;
AB= s1/10000
MES= s2/10000
ACI= inter/10000
prop= c/10000

```

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CHAPTER 6

Marshall-Olkin Exponentiated Generalized Fréchet Distribution and its Applications

6.1 Introduction

In 1927 a French mathematician Maurice Fréchet has introduced Fréchet distribution. It is a special case of generalized extreme value distribution and is also known as type II extreme value distribution, which is equivalent to taking the reciprocal of values from a standard Weibull distribution. Extreme Value distributions, are widely used in risk management, finance, insurance, economics, hydrology, material sciences, telecommunications and many other industries dealing with extreme events. The Fréchet distribution is useful for modeling and analysis of several extreme events ranging from accelerated life testing to earthquakes, floods, rainfall, sea currents and wind speeds, etc. More information about the Fréchet distribution can be found in Kotz and Nadarajah(2000), Coles (2001) and Johnson et al. (2004), Nadarajah and Kotz(2008), Mubarak(2012), Harlow (2002) etc.

Abd-Elfattah et al.(2010) discussed goodness of fit tests for generalized Fréchet distribution.

Nadarajah and Kotz(2003) introduced the exponentiated Fréchet distribution with distribution function

$$F(x) = 1 - (1 - \exp\{-(\frac{\sigma}{x})^\lambda\})^\alpha;$$

$x > 0, \alpha > 0, \beta > 0, \lambda > 0, \sigma > 0$. For $\alpha = 1$, the exponentiated Fréchet distribution becomes the Fréchet distribution with parameters λ and σ . Abd-Elfattah and Omima (2009) discussed estimation of parameters of the generalized Fréchet distribution. Nadarajah and Gupta (2004) introduced the Beta Fréchet distribution with the distribution function

$$F(x) = \frac{1}{B(a, b)} \int_0^{e^{-(\frac{\sigma}{x})^\lambda}} w^{a-1}(1-w)^{b-1} dw; x, \sigma, \lambda, a, b > 0.$$

The Beta Fréchet distribution generalizes some well known distributions. For $a = 1$, we obtain the exponentiated Fréchet distribution with parameters σ, λ and $\alpha = b$. For $a = 1$ and $b = 1$ we obtain the Fréchet distribution with parameters σ and λ .

Recently, Krishna et al.(2013 a) introduced Marshall-Olkin Fréchet distribution with survival function given by

$$\bar{G}(x) = \frac{\theta(1 - \exp\{-(\frac{\sigma}{x})^\lambda\})}{\theta + (1 - \theta)\exp\{-(\frac{\sigma}{x})^\lambda\}}$$

where $x > 0, \theta > 0, \lambda > 0$ and $\sigma > 0$. Krishna et al.(2013 b) also discussed the applications of Marshall-Olkin Fréchet distribution. Mahmoud et al.(2013) introduced the transmuted Fréchet distribution. Here we consider the EG class of distributions corre-

sponding to Fréchet distribution and it is extended to the Marshall-Olkin exponentiated generalized Fréchet distribution.

This chapter concentrates on Marshall-Olkin Exponentiated Generalized Fréchet Distribution and its Applications. In section 3.2 we discuss the important properties of the Exponentiated generalized Fréchet distribution. Marshall-Olkin Exponentiated generalized Fréchet Distribution and its properties are discussed in section 3.3. In section 3.4 we consider the quantiles and distribution of order statistics. In section 3.5 the maximum likelihood estimates are obtained and the results are applied to a real data set to compare the new distribution with Exponentiated generalized Fréchet distribution. Reliability of a system following Marshall-Olkin extended Fréchet distribution under stress-strength model is estimated in section 3.6. Its validity is examined using average bias and average mean square error calculated from the simulated values. Simulation studies are conducted to compute the average length of the asymptotic 95% confidence intervals and coverage probability.

6.2 Exponentiated Generalized Fréchet Distribution

The c.d.f. of the Fréchet distribution is $G(x) = \exp\{-\left(\frac{\sigma}{x}\right)^\lambda\}$ where $\sigma, \lambda > 0$. Then we define the Exponentiated Generalized Fréchet (EGF) with cumulative distribution by

$$F(x) = [1 - (1 - \exp\{-\left(\frac{\sigma}{x}\right)^\lambda\})^\alpha]^\beta \quad (6.2.1)$$

where $x > 0, \alpha > 0, \beta > 0, \lambda > 0$ and $\sigma > 0$. The EGF density can be obtained from (1.3.4) as

$$f(x) = \alpha\beta\lambda\sigma^\lambda x^{-(\lambda+1)} \exp\{-\left(\frac{\sigma}{x}\right)^\lambda\} \left(1 - \exp\{-\left(\frac{\sigma}{x}\right)^\lambda\}\right)^{\alpha-1} \left[1 - \left(1 - \exp\{-\left(\frac{\sigma}{x}\right)^\lambda\}\right)^\alpha\right]^{\beta-1}$$

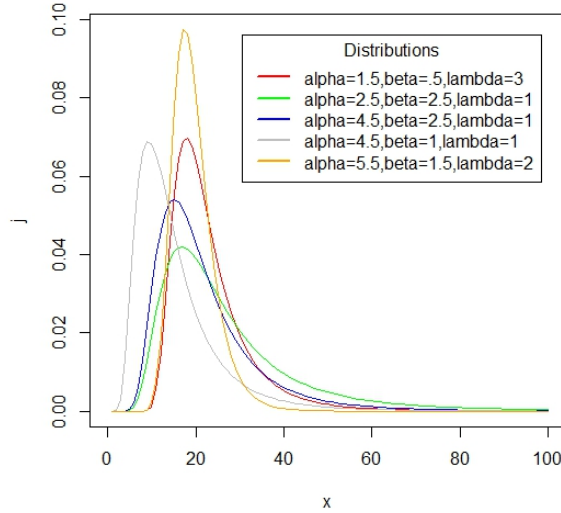


Figure 6.1: The p.d.f. of EGF distribution

Here when $\beta = 1$, the distribution reduces to the Exponentiated Fréchet distribution and when $\alpha = 1, \beta = 1$, it reduces to the standard Fréchet distribution. The survival function and hazard rate function of the EGF distribution is given by

$$\bar{F}(x) = 1 - [1 - (1 - \exp\{-\left(\frac{\sigma}{x}\right)^\lambda\})^\alpha]^\beta \quad (6.2.2)$$

and

$$r(x) = \alpha\beta\lambda\sigma^\lambda x^{-(\lambda+1)} \exp\{-\left(\frac{\sigma}{x}\right)^\lambda\} \left(1 - \exp\{-\left(\frac{\sigma}{x}\right)^\lambda\}\right)^{\alpha-1} \frac{[1 - (1 - \exp\{-\left(\frac{\sigma}{x}\right)^\lambda\})^\alpha]^{\beta-1}}{1 - [1 - (1 - \exp\{-\left(\frac{\sigma}{x}\right)^\lambda\})^\alpha]^\beta}$$

Figure 6.1 shows the p.d.f. of EGF distribution for different values of parameters.

Figure 6.2 shows the hazard function of EGF distribution for different values of param-

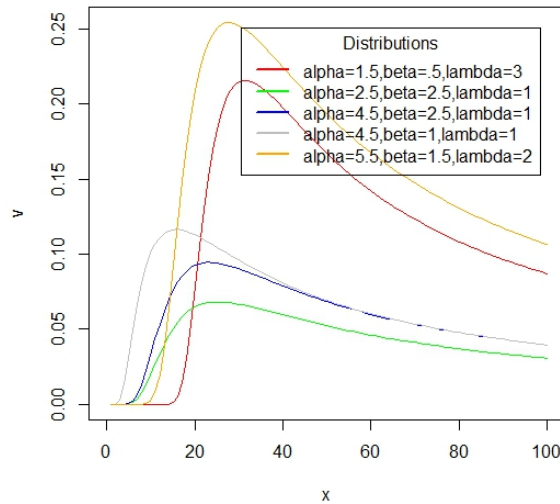


Figure 6.2: Hazard rate function of EGF distribution

eters.

The p^{th} quantile function x_p of the EGF distribution, which is the inverse of the distribution function $F(x_p) = p$ is given by

$$x_p = \left[\frac{-\sigma^\lambda}{\log[1 - (1 - p)^{\frac{1}{\beta}}]^{\frac{1}{\alpha}}} \right]^{\frac{1}{\lambda}}$$

6.3 Marshall-Olkin Exponentiated Generalized Fréchet Distribution

Consider the survival function of exponentiated generalized Fréchet distribution $\bar{F}(x) = 1 - [1 - (1 - \exp\{-\left(\frac{\sigma}{x}\right)^\lambda\})^\alpha]^\beta$. In this section we introduce the Marshall-Olkin Exponentiated Generalized Fréchet Distribution using Marshall-Olkin techniques. It is denoted by MOEGF

distribution. The survival function of MOEGF distribution is given by

$$\bar{G}(x) = \frac{\theta [1 - [1 - (1 - \exp\{-(\frac{\sigma}{x})^\lambda\})^\alpha]^\beta]}{1 - (1 - \theta) [1 - [1 - (1 - \exp\{-(\frac{\sigma}{x})^\lambda\})^\alpha]^\beta]} \quad (6.3.1)$$

Then the corresponding probability density function is given by

$$g(x) = \theta \alpha \beta \lambda \sigma^\lambda x^{-(\lambda+1)} \exp\{-(\frac{\sigma}{x})^\lambda\} \left(1 - \exp\{-(\frac{\sigma}{x})^\lambda\}\right)^{\alpha-1} \quad (6.3.2)$$

$$\frac{[1 - (1 - \exp\{-(\frac{\sigma}{x})^\lambda\})^\alpha]^{\beta-1}}{(1 - \bar{\theta} + \bar{\theta}[1 - (1 - \exp\{-(\frac{\sigma}{x})^\lambda\})^\alpha]^\beta)^2} \quad (6.3.3)$$

$\alpha > 0, \beta > 0, \lambda > 0, \sigma > 0, \theta > 0.$

The cumulative distribution function is given by

$$G(x) = \frac{[1 - (1 - \exp\{-(\frac{\sigma}{x})^\lambda\})^\alpha]^\beta}{\theta + (1 - \theta)[1 - (1 - \exp\{-(\frac{\sigma}{x})^\lambda\})^\alpha]^\beta}$$

and the hazard rate function is given by

$$h(x) = \frac{\alpha \beta \lambda \sigma^\lambda x^{-(\lambda+1)} \exp\{-(\frac{\sigma}{x})^\lambda\} [A(x)]^{\alpha-1} [1 - [A(x)]^\alpha]^{\beta-1}}{K(x)[1 - \bar{\alpha}K(x)]}$$

where $A(x) = (1 - \exp\{-(\frac{\sigma}{x})^\lambda\})$ and

$$K(x) = [1 - [1 - (1 - \exp\{-(\frac{\sigma}{x})^\lambda\})^\alpha]^\beta]$$

Figure 6.3 shows the pdf of MOEGF distribution for different values of parameters.

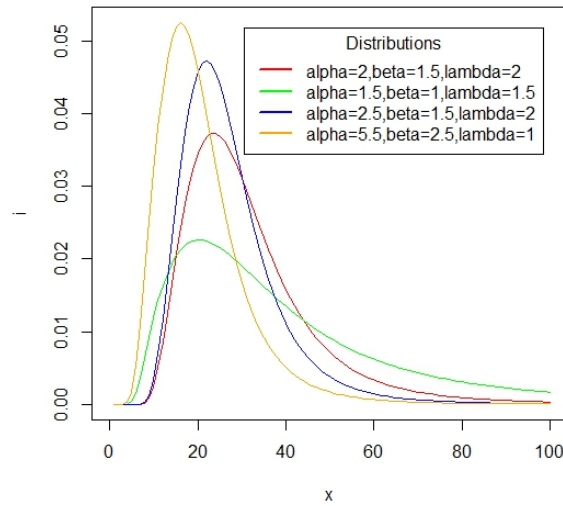


Figure 6.3: The p.d.f. of MOEGF distribution for different values of parameters

6.4 Quantiles and Order statistics

The p^{th} quantile function of the distribution, which is the inverse of the distribution function $F(x_p) = p$, is given by

$$x = \frac{-1}{\sigma} \left[\log \left\{ \left[1 - \frac{p\theta}{1 - p(1 - \theta)} \right]^{\frac{1}{\beta}} \right\}^{\frac{1}{\alpha}} \right]^{\frac{1}{\lambda}}$$

Let X_1, X_2, \dots, X_n be a random sample taken from the MOEGF distribution and $X_{1:n}, X_{2:n}, \dots, X_{n:n}$ be the corresponding order statistics. The survival function of MOEGF distribution is given by (6.3.1). Then the c.d.f. of the first order statistic $X_{1:n}$

is given by

$$\begin{aligned} G_{1:n}(x) &= 1 - (\bar{G}(x))^n \\ &= 1 - \left[\frac{\theta [1 - [1 - (1 - \exp\{-(\frac{\sigma}{x})^\lambda\})^\alpha]^\beta]}{1 - (1 - \theta) [1 - [1 - (1 - \exp\{-(\frac{\sigma}{x})^\lambda\})^\alpha]^\beta]} \right]^n \end{aligned}$$

The c.d.f. of the n^{th} order statistic $X_{n:n}$ is given by

$$\begin{aligned} G_{n:n}(x) &= [1 - \bar{G}(x)]^n \\ &= \left\{ \frac{[1 - (1 - \exp\{-(\frac{\sigma}{x})^\lambda\})^\alpha]^\beta}{\theta + (1 - \theta)[1 - (1 - \exp\{-(\frac{\sigma}{x})^\lambda\})^\alpha]^\beta} \right\}^n \end{aligned}$$

The probability density function $g_{i:n}(x)$ of the i^{th} order statistics $X_{i:n}$ is given by

$$\begin{aligned} g_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!} \theta \alpha \beta \lambda \sigma^\lambda x^{-(\lambda+1)} \exp\{-(\frac{\sigma}{x})^\lambda\} \left(1 - \exp\{-(\frac{\sigma}{x})^\lambda\}\right)^{\alpha-1} \\ &\quad \frac{[1 - (1 - \exp\{-(\frac{\sigma}{x})^\lambda\})^\alpha]^{\beta-1}}{(1 - \bar{\theta} + \bar{\theta}[1 - (1 - \exp\{-(\frac{\sigma}{x})^\lambda\})^\alpha]^\beta)^2} \\ &\quad \left\{ \frac{[1 - (1 - \exp\{-(\frac{\sigma}{x})^\lambda\})^\alpha]^\beta}{\theta + (1 - \theta)[1 - (1 - \exp\{-(\frac{\sigma}{x})^\lambda\})^\alpha]^\beta} \right\}^{i-1} \\ &\quad \left\{ \frac{\theta [1 - [1 - (1 - \exp\{-(\frac{\sigma}{x})^\lambda\})^\alpha]^\beta]}{1 - (1 - \theta) [1 - [1 - (1 - \exp\{-(\frac{\sigma}{x})^\lambda\})^\alpha]^\beta]} \right\}^{n-i} \end{aligned}$$

6.5 Estimation of Parameters

In this section we consider maximum likelihood estimates of the parameters with respect to a given sample (x_1, x_2, \dots, x_n) . Then the log likelihood function is given by

$$\begin{aligned} \log L(\alpha, \beta, \sigma, \lambda, \theta) = & n [\log \theta + \log \alpha + \log \beta + \log \lambda + \lambda \log \sigma] - \\ & (\lambda + 1) \sum_{i=1}^n \log x_i - \lambda \sum_{i=1}^n \left(\frac{\sigma}{x_i}\right) + (\alpha - 1) \sum_{i=1}^n \log[1 - \exp\{-(\frac{\sigma}{x_i})^\lambda\}] + \\ & (\beta - 1) \sum_{i=1}^n [1 - (1 - \exp\{-(\frac{\sigma}{x_i})^\lambda\})^\alpha] - \\ & 2 \sum_{i=1}^n \log \left\{ 1 - \bar{\theta} + \bar{\theta} [1 - (1 - \exp\{-(\frac{\sigma}{x_i})^\lambda\})^\alpha]^\beta \right\} \end{aligned}$$

The partial derivative of the log likelihood functions with respect to the parameters are

$$\begin{aligned} \frac{\partial \log L}{\partial \alpha} = & \frac{n}{\alpha} + \sum_{i=1}^n \log(1 - \exp\{-(\frac{\sigma}{x_i})^\lambda\}) - (\beta - 1) \sum_{i=1}^n \log(1 - \exp\{-(\frac{\sigma}{x_i})^\lambda\}) \\ & \frac{(1 - \exp\{-(\frac{\sigma}{x_i})^\lambda\})^\alpha}{1 - (1 - \exp\{-(\frac{\sigma}{x_i})^\lambda\})^\alpha} + 2\theta\beta \sum_{i=1}^n (1 - \exp\{-(\frac{\sigma}{x_i})^\lambda\})^\alpha \\ & \log(1 - \exp\{-(\frac{\sigma}{x_i})^\lambda\}) \frac{[1 - (1 - \exp\{-(\frac{\sigma}{x_i})^\lambda\})^\alpha]^{\beta-1}}{\{1 - \bar{\theta} + \bar{\theta} [1 - (1 - \exp\{-(\frac{\sigma}{x_i})^\lambda\})^\alpha]^\beta\}} \end{aligned}$$

$$\begin{aligned} \frac{\partial \log L}{\partial \beta} = & \frac{n}{\beta} + \sum_{i=1}^n \log[1 - (1 - \exp\{-(\frac{\sigma}{x_i})^\lambda\})^\alpha] - \\ & 2\theta \sum_{i=1}^n \log[1 - (1 - \exp\{-(\frac{\sigma}{x_i})^\lambda\})^\alpha] \frac{[1 - (1 - \exp\{-(\frac{\sigma}{x_i})^\lambda\})^\alpha]^\beta}{\{1 - \bar{\theta} + \bar{\theta} [1 - (1 - \exp\{-(\frac{\sigma}{x_i})^\lambda\})^\alpha]^\beta\}} \end{aligned}$$

$$\begin{aligned} \frac{\partial \log L}{\partial \sigma} &= \frac{n\lambda}{\sigma} + \lambda \sum_{i=1}^n \frac{1}{x_i} + (\alpha - 1)\lambda\sigma^{\lambda-1} \sum_{i=1}^n \frac{(\frac{1}{x_i})^\lambda \exp\{-\frac{(\sigma}{x_i})^\lambda\}}{(1 - \exp\{-\frac{(\sigma}{x_i})^\lambda\})} \\ &- (\beta - 1)\alpha\lambda\sigma^{\lambda-1} \sum_{i=1}^n \exp\{-\frac{(\sigma}{x_i})^\lambda\} (\frac{1}{x_i})^\lambda \frac{(1 - \exp\{-\frac{(\sigma}{x_i})^\lambda\})^{\alpha-1}}{[1 - (1 - \exp\{-\frac{(\sigma}{x_i})^\lambda\})^\alpha]} \\ &- 2\alpha\beta\theta\lambda\sigma^{\lambda-1} \sum_{i=1}^n (\frac{1}{x_i})^\lambda \exp\{-\frac{(\sigma}{x_i})^\lambda\} (1 - \exp\{-\frac{(\sigma}{x_i})^\lambda\})^{\alpha-1} \\ &\quad \frac{[1 - (1 - \exp\{-\frac{(\sigma}{x_i})^\lambda\})^\alpha]^{\beta-1}}{\left\{1 - \bar{\theta} + \bar{\theta}[1 - (1 - \exp\{-\frac{(\sigma}{x_i})^\lambda\})^\alpha]^\beta\right\}} \end{aligned}$$

$$\begin{aligned} \frac{\partial \log L}{\partial \lambda} &= n \left(\frac{1}{\lambda} + \log \sigma \right) - \sum_{i=1}^n \log x_i - \sigma \sum_{i=1}^n \frac{1}{x_i} + (\alpha - 1)\sigma \\ &\quad \sum_{i=1}^n \frac{(\frac{1}{x_i}) \exp\{-\frac{(\sigma}{x_i})^\lambda\} \log(\frac{\sigma}{x_i})}{(1 - \exp\{-\frac{(\sigma}{x_i})^\lambda\})} - (\beta - 1)\alpha\sigma^\lambda \\ &\quad \sum_{i=1}^n (\frac{1}{x_i})^\lambda \exp\{-\frac{(\sigma}{x_i})^\lambda\} \log(\frac{\sigma}{x_i}) \frac{(1 - \exp\{-\frac{(\sigma}{x_i})^\lambda\})^{\alpha-1}}{[1 - (1 - \exp\{-\frac{(\sigma}{x_i})^\lambda\})^\alpha]} \\ &\quad - 2\alpha\beta\theta\sigma^\lambda \sum_{i=1}^n (\frac{1}{x_i})^\lambda \exp\{-\frac{(\sigma}{x_i})^\lambda\} \log(\frac{\sigma}{x_i}) \\ &\quad (1 - \exp\{-\frac{(\sigma}{x_i})^\lambda\})^{\alpha-1} \frac{[1 - (1 - \exp\{-\frac{(\sigma}{x_i})^\lambda\})^\alpha]^{\beta-1}}{\left\{1 - \bar{\theta} + \bar{\theta}[1 - (1 - \exp\{-\frac{(\sigma}{x_i})^\lambda\})^\alpha]^\beta\right\}} \end{aligned}$$

$$\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} - 2 \sum_{i=1}^n \frac{[1 - (1 - \exp\{-\frac{(\sigma}{x_i})^\lambda\})^\alpha]^\beta}{\left\{1 - \bar{\theta} + \bar{\theta}[1 - (1 - \exp\{-\frac{(\sigma}{x_i})^\lambda\})^\alpha]^\beta\right\}}$$

The maximum likelihood estimates can be obtained by solving the equations

$$\frac{\partial \log L}{\partial \alpha} = 0, \frac{\partial \log L}{\partial \beta} = 0, \frac{\partial \log L}{\partial \sigma} = 0, \frac{\partial \log L}{\partial \lambda} = 0, \frac{\partial \log L}{\partial \theta} = 0$$

The equations can be solved using nlm package in R software.

Table 6.1: Summary of fitting for the MOEGF and exponentiated generalized Fréchet distribution.

Model	Parameter	Estimates	-Log-likelihood	K-S statistic
EGF	α	0.0828	416.6228	0.8078
	β	0.0849		
	λ	0.0718		
	σ	0.0245		
MOEGF	α	0.0758	304.626	0.6975
	β	0.0763		
	λ	0.0849		
	σ	0.0508		
	θ	0.0654		

6.5.1 Data Analysis

In this section we analyze some data sets and compare Marshall-Olkin exponentiated generalized Fréchet distribution with the exponentiated generalized Fréchet distribution. We consider the data from Lawless(1986). The data given here arose in tests on endurance of deep groove ball bearings. The data are the number of million revolutions before failure for each of the 23 ball bearings in the life test and they are

17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.80, 51.84, 51.96, 54.12, 55.56, 67.80,68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40.

We estimate the unknown parameters of the distribution by the method of maximum likelihood estimation. Also, we draw the P-P plots and Q-Q plots for fitted distributions and are presented in Figure 6.4 and in Figure 6.5. We can see that the Marshall-Olkin exponentiated generalized Fréchet distribution is a good fit as compared to exponentiated generalized Fréchet distribution.

6.6 Stress-Strength Analysis

In this section we consider the stress-strength reliability $R = P(X < Y)$, where X represents stress and Y represents the strength. Gupta et al. (2010) obtained various results on the MO family in the context of reliability modeling and survival analysis.

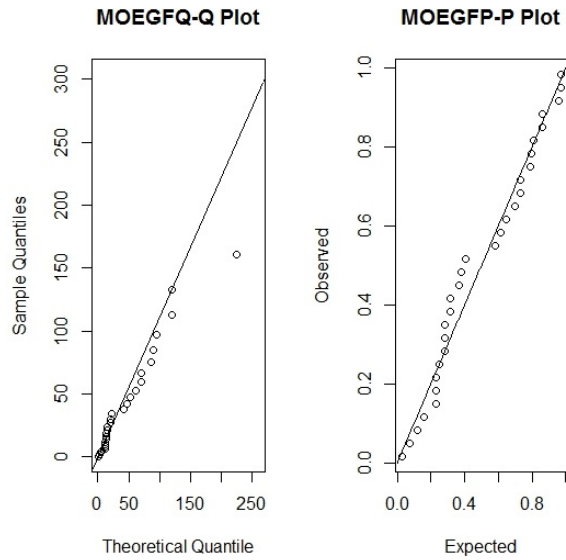


Figure 6.4: QQ and PP plot for MOEGF Distribution

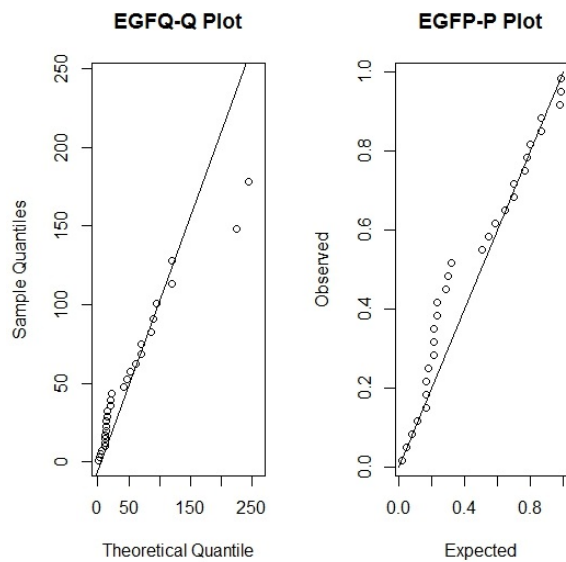


Figure 6.5: QQ and PP plot for EGF Distribution

$$\begin{aligned} P(X < Y) &= \int_{-\infty}^{\infty} P(Y > X/X = x)g_X(x)dx \\ &= \frac{\frac{\alpha_1}{\alpha_2}}{(\frac{\alpha_1}{\alpha_2} - 1)^2} \left[-\log \frac{\alpha_1}{\alpha_2} + \frac{\alpha_1}{\alpha_2} - 1 \right] \end{aligned}$$

Consider the p.d.f. of Marshall-Olkin Exponentiated Generalized Frèchet distribution given by

$$g(x) = \theta\alpha\beta\lambda\sigma^\lambda x^{-(\lambda+1)} \exp\left\{-\left(\frac{\sigma}{x}\right)^\lambda\right\} \left(1 - \exp\left\{-\left(\frac{\sigma}{x}\right)^\lambda\right\}\right)^{\alpha-1} \frac{\left[1 - \left(1 - \exp\left\{-\left(\frac{\sigma}{x}\right)^\lambda\right\}\right)^\alpha\right]^{\beta-1}}{\left(1 - \bar{\theta} + \bar{\theta}\left[1 - \left(1 - \exp\left\{-\left(\frac{\sigma}{x}\right)^\lambda\right\}\right]^\alpha\right)^\beta\right)^2}$$

$\alpha > 0, \beta > 0, \lambda > 0, \theta > 0, \sigma > 0$.

Let (x_1, x_2, \dots, x_m) and (y_1, y_2, \dots, y_n) be two independent random samples of sizes m and n from Marshall-Olkin Exponentiated Generalized Frèchet distribution with tilt parameters α_1 and α_2 respectively, and common unknown parameters β, λ and θ . The log likelihood function is given by

$$L(\alpha_1, \alpha_2, \beta, \lambda, \theta) = \sum_{i=1}^m \log g(x_i; \alpha_1, \beta, \lambda, \theta, \sigma) + \sum_{j=1}^n \log g(y_j; \alpha_2, \beta, \lambda, \theta, \sigma)$$

The maximum likelihood estimates of the unknown parameters α_1, α_2 are the solutions of non linear equations $\frac{\partial L}{\partial \alpha_1} = 0$ and $\frac{\partial L}{\partial \alpha_2} = 0$ respectively.

6.6.1 Simulation Study

We generate $N = 10000$ values of X and Y observations from Marshall-Olkin Exponentiated Generalized Frèchet distribution with parameters $\alpha_1, \beta, \lambda, \theta, \sigma$ and $\alpha_2, \beta, \lambda, \theta, \sigma$ respectively. The combinations of samples of sizes $m = 20, 25, 30$ and $n = 20, 25, 30$ are

considered. The estimates of α_1 and α_2 are obtained from each sample to obtain \hat{R} . The validity of the estimate of R is examined as in section 5.6. The average bias and average mean square error of the simulated estimates of R for various values of parameters are given in Table 6.2. The average confidence length and coverage probability of the simulated estimates are given in Table 6.3.

6.7 Conclusion

In this chapter we have introduced a new distribution namely, Marshall-Olkin Exponentiated Generalized Fréchet Distribution and its properties are discussed. We analyze a real data set and compare MOEGF distribution with the exponentiated generalized Fréchet distribution. We conclude that the MOEGF distribution is a better fit as compared to exponentiated generalized Fréchet distribution. The results are given in Table 6.1. Estimation of stress-strength reliability is also done. The average bias and average mean square error of the simulated estimates of R for various values of parameters are given in Table 6.2. The average confidence length and coverage probability of various values of parameters are given in Table 6.3. The R program developed is given as Appendix.

6.8 Appendix

```
#MOEGF DISTRIBUTION
# Probability density function
dMOEGF<-function(x,alpha,beta,lambda,sigma,theta) {
  temp <- exp(-(sigma/x)^lambda)
  theta*alpha*beta*lambda*sigma^lambda*x^(lambda+1)*temp*(1-temp)^(alpha-1)*(1-(1-temp)^alpha)^(beta-1)
  /(1-(1-theta)+(1-theta)*(1-(1-temp)^alpha)^beta)^2
}
# Distribution function
pMOEGF<-function(x,alpha,beta,lambda,sigma,theta) {
  temp <- -(sigma/x)^lambda
  temp <- (1-exp(temp))^alpha
  theta*(1-(1-temp)^beta)/(1-(1-theta)*(1-(1-temp)^beta))
}
# Quantile function
qMOEGF<-function(p,alpha,beta,lambda,sigma,theta) {
  -1/sigma*(log((1-p*theta/(1-p*(1-theta)))^(1/beta))^(1/alpha))^(1/lambda)
}
```

Table 6.2: Average bias and average MSE of the simulated estimates of R for $\beta = 4, \lambda = 3, \sigma = 2$ and $\theta = 2$

(α_1, α_2)								
(m,n)	Average bias (<i>b</i>)				Average Mean Square Error (AMSE)			
	(0.8,0.9)	(1.2,1.4)	(1.5,1.9)	(2,1.5)	(0.8,0.9)	(1.2,1.4)	(1.5,1.9)	(2,1.5)
(20,20)	0.0378	0.0509	0.0771	-0.0938	0.0037	0.0051	0.0087	0.0115
(20,25)	0.0384	0.0496	0.0767	-0.0934	0.0037	0.0051	0.0086	0.0114
(20,30)	0.0380	0.0508	0.0772	-0.0924	0.0037	0.0052	0.0086	0.0112
(25,20)	0.0395	0.0530	0.0788	-0.0920	0.0036	0.0051	0.0086	0.0109
(25,25)	0.0406	0.0527	0.0803	-0.0909	0.0037	0.0051	0.0089	0.0108
(25,30)	0.0402	0.0521	0.0801	-0.0922	0.0037	0.0050	0.0088	0.0110
(30,20)	0.0415	0.0541	0.0810	-0.0895	0.0036	0.0051	0.0088	0.0102
(30,25)	0.0421	0.0539	0.0815	-0.0897	0.0037	0.0051	0.0089	0.0103
(30,30)	0.0408	0.0539	0.0813	-0.0894	0.0035	0.0051	0.0088	0.0102

Table 6.3: Average confidence length and coverage probability of the simulated 95% percentage confidence intervals of R for $\beta = 4, \lambda = 3, \sigma = 2$ and $\theta = 2$

(α_1, α_2)								
(m,n)	Average Confidence Length				Coverage Probability			
	(0.8,0.9)	(1.2,1.4)	(1.5,1.9)	(2,1.5)	(0.8,0.9)	(1.2,1.4)	(1.5,1.9)	(2,1.5)
(20,20)	0.3544	0.3537	0.3525	0.3517	0.9963	0.9896	0.9630	0.9311
(20,25)	0.3363	0.3356	0.3344	0.3337	0.9934	0.9849	0.9486	0.9079
(20,30)	0.3236	0.3229	0.3218	0.3212	0.9922	0.9791	0.9379	0.8910
(25,20)	0.3335	0.3357	0.3346	0.3341	0.9948	0.9871	0.9544	0.9283
(25,25)	0.3172	0.3166	0.3153	0.3150	0.9928	0.9815	0.9304	0.8947
(25,30)	0.3037	0.3031	0.3019	0.3015	0.9899	0.9760	0.9153	0.8674
(30,20)	0.3238	0.3232	0.3217	0.3220	0.9945	0.9845	0.9420	0.9216
(30,25)	0.3038	0.3032	0.3020	0.3020	0.9891	0.9745	0.9148	0.8893
(30,30)	0.2869	0.2891	0.2880	0.2880	0.9869	0.9668	0.8941	0.8961

```

%# Random generation
rMOEGF<-function(alpha,beta,lambda,sigma,theta,nobs) {
  varSample<-double(nobs)
  for (i in 1:nobs) {
    p<-runif(1)
    varSample[i]<- -1/sigma*(log((1-p*theta/(1-p*(1-theta))))^(1/beta))^(1/alpha))^(1/lambda)
  }
  varSample
}
# Log-likelihood function
logLikelihoodMOEGF<-function(x) {
  alpha<-x[1]
  beta<-x[2]
  lambda<-x[3]
  sigma<-x[4]
  theta<-x[5]
  nobs<-length(y);
  temp <- 1-exp(-(sigma/x)^lambda)
  nobs*log(theta)+nobs*log(alpha)+nobs*log(beta)+nobs*log(lambda)+nobs*lambda*log(sigma)
  -(lambda+1)*sum(log(x))-lambda*sum(sigma/x)+(alpha-1)*sum(log(temp))+(beta-1)*sum(1-temp^alpha)
  -2*sum(log(1-(1-theta)+(1-theta)*(1-temp^alpha)^beta))
}
gradientMOEGF <- function(x) {
  alpha<-x[1]
  beta<-x[2]
  lambda<-x[3]
  sigma<-x[4]
  theta<-x[5]
  nobs<-length(y);
  temp <- exp(-(sigma/x)^lambda)
  der1 <- n/alpha+sum(log(1-temp))-(beta-1)*sum(log(1-temp)*(1-temp)^alpha/(1-(1-temp)^alpha))+2*theta*beta*
  sum((1-temp)^alpha*log(1-temp)*(1-(1-temp)^alpha)^(beta-1)/(1-(1-theta)+(1-theta)*(1-(1-temp)^alpha)^beta))
  der2 <- n/beta+sum(log(1-(1-temp)^alpha))-2*theta*sum(log(1-(1-temp)^alpha)*(1-(1-temp)^alpha)^beta
  /(1-(1-theta)+(1-theta)*(1-(1-temp)^alpha)^beta))
  der3 <- n*(1/lambda+log(sigma))-sum(log(x))-sigma*sum(1/x)+(alpha-1)*sigma*sum((1/x)*temp*log(sigma/x)/(1-temp))
  -(beta-1)*alpha*sigma^lambda*sum(1/x^lambda*temp*log(sigma/x)*(1-temp)^(alpha-1)/(1-(1-temp)^alpha))-2*alpha*beta
  *theta*sigma^lambda*sum(1/x^lambda*temp*log(sigma/x)*(1-temp)^(alpha-1)*(1-(1-temp)^alpha)^(beta-1)/
  (1-(1-theta)+(1-theta)*(1-(1-temp)^alpha)^beta))
  der4 <- n*lambda/sigma+lambda*sum(1/x)+(alpha-1)*lambda*sigma^(lambda-1)*sum(1/x^lambda*temp/(1-temp))-
  (beta-1)*alpha*lambda*sigma^(lambda-1)*sum(temp/x^lambda*(1-temp)^(alpha-1)/(1-(1-temp)^alpha))-2*alpha*
  beta*theta*lambda*sigma^(lambda-1)*sum(1/x^lambda*temp*(1-temp)^(alpha-1)*(1-(1-temp)^alpha)^(beta-1)
  /(1-(1-theta)+(1-theta)*(1-(1-temp)^alpha)^beta))
  der5 <- n/theta-2*sum((1-(1-temp)^alpha)^beta/(1-(1-theta)+(1-theta)*(1-(1-temp)^alpha)^beta))
}

```

```

    c(der1,der2,der3,der4,der5)
}
# We minimize the following function
functionMOEGF<-function(x) {
  res<--logLikelihoodMOEGF(x);
  # attr(res,"gradient") <- -gradientMOEGF(x);
  res
}
# QQ plot
qqMOEGF<-function(y,alpha,beta,lambda,sigma,theta) {
  nn<-length(y);
  x<-qMOEGF(ppoints(nn),alpha,beta,lambda,sigma,theta)[order(order(y))];
  plot(y,x,main="MOEGF Q-Q Plot",xlab="Theoretical Quantile",ylab="Sample Quantiles");
  z<-quantile(y,c(0.25,0.75));
  xx<-qMOEGF(c(0.25,0.75),alpha,beta,lambda,sigma,theta);
  slope<-diff(z)/diff(xx);
  int<-z[1]-slope*xx[1];
  abline(int,slope)
}
# PP plot
ppMOEGF<-function(y,alpha,beta,lambda,sigma,theta) {
  nn<-length(y);
  yvar<-pMOEGF(sort(y),alpha,beta,lambda,sigma,theta);
  xvar<-1:nn;
  x<-(xvar-0.5)/nn;
  plot(yvar,x,main="MOEGF P-P Plot",xlab="Expected",ylab="Observed");
  curve(1*x,0,1,add=T)
}
#EGF DISTRIBUTION:
# Probability density function
dEGF<-function(x,alpha,beta,lambda,sigma) {
  temp <- exp(-(sigma/x)^lambda)
  alpha*beta*lambda*sigma^lambda*x^(lambda+1)*temp*(1-temp)^(alpha-1)*(1-(1-temp)^alpha)^(beta-1)
}
# Distribution function
pEGF<-function(x,alpha,beta,lambda,sigma) {
  temp <- -(sigma/x)^lambda
  temp <- (1-exp(temp))^alpha
  (1-(1-temp)^beta)
}
# Quantile function
qEGF<-function(p,alpha,beta,lambda,sigma) {
  (-sigma^lambda/log(1-(1-p)^(1/beta))^(1/alpha) )^(1/lambda)
}

```

```

}
# Random generation
rEGF<-function(alpha,beta,lambda,sigma,nobs) {
  varSample<-double(nobs)
  for (i in 1:nobs) {
    p<-runif(1)
    varSample[i]<-(-sigma^lambda/log(1-(1-p)^(1/beta))^(1/alpha) )^(1/lambda)
  }
  varSample
}
# Log-likelihood function
logLikelihoodEGF<-function(x) {
  alpha<-x[1]
  beta<-x[2]
  lambda<-x[3]
  sigma<-x[4]
  nobs<-length(y);
  temp <- 1-exp(-(sigma/x)^lambda)
  nobs*log(alpha)+nobs*log(beta)+nobs*log(lambda)+nobs*lambda*log(sigma)-
  (lambda+1)*sum(log(x))-lambda*sum(sigma/x)+(alpha-1)*sum(log(temp))+
  (beta-1)*sum(1-temp^alpha)
}
#gradientEGF <- function(x) {
  alpha<-x[1]
  beta<-x[2]
  lambda<-x[3]
  sigma<-x[4]
  nobs<-length(y);
  temp <- exp(-(sigma/x)^lambda)
  der1 <- n/alpha+sum(log(1-temp))-(beta-1)*sum(log(1-temp)*(1-temp)^alpha/(1-(1-temp)^alpha))
  der2 <- n/beta+sum(log(1-(1-temp)^alpha))
  der3 <- n*(1/lambda+log(sigma))-sum(log(x))-sigma*sum(1/x)+(alpha-1)*sigma*sum((1/x)*temp*log(sigma/x)/(1-temp))
  -(beta-1)*alpha*sigma^lambda*sum(1/x^lambda*temp*log(sigma/x)*(1-temp)^(alpha-1)/(1-(1-temp)^alpha))
  der4 <-n*lambda/sigma+lambda*sum(1/x)+(alpha-1)*lambda*sigma^(lambda-1)*sum(1/x^lambda*temp/
  (1-temp))-(beta-1)*alpha*lambda*sigma^(lambda-1)*sum(temp/x^lambda*(1-temp)^(alpha-1)/(1-(1-temp)^alpha))
  c(der1,der2,der3,der4)
}
# We minimize the following function
functionEGF<-function(x) {
  res<--logLikelihoodEGF(x);
  # attr(res,"gradient") <- -gradientMOEGF(x);
  res
}
-loglikelihoodEGF(x);

```

```
# QQ plot
qqEGF<-function(y,alpha,beta,lambda,sigma) {
  nn<-length(y);
  x<-qEGF(ppoints(nn),alpha,beta,lambda,sigma)[order(order(y))];
  plot(y,x,main="EGF Q-Q Plot",xlab="Theoretical Quantile",ylab="Sample Quantiles");
  z<-quantile(y,c(0.25,0.75));
  xx<-qEGF(c(0.25,0.75),alpha,beta,lambda,sigma);
  slope<-diff(z)/diff(xx);
  int<-z[1]-slope*xx[1];
  abline(int,slope)
}
# PP plot
ppEGF<-function(y,alpha,beta,lambda,sigma) {
  nn<-length(y);
  yvar<-pEGF(sort(y),alpha,beta,lambda,sigma);
  xvar<-1:nn;
  x<-(xvar-0.5)/nn;
  plot(yvar,x,main="EGF P-P Plot",xlab="Expected",ylab="Observed");
  curve(1*x,0,1,add=T)
}
}
```

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CHAPTER 7

Exponentiated Marshall-Olkin Exponential and Weibull Distributions

7.1 Introduction

Exponential and Weibull distribution play a central role in reliability theory and survival analysis. Marshall and Olkin (1997) introduced a new family of distributions by adding a parameter to obtain new families of distributions which are more flexible and represent a wide range of behavior than the original distributions. Here we consider the exponentiated Marshall-Olkin family of distributions which can be regarded as Gamma compounding models. Many authors have proposed various univariate distributions belonging to Marshall-Olkin family of distributions. Ghitany et al.(2005) introduced Marshall-Olkin extended Weibull which can be obtained as a compound distribution with mixing exponential distribution. Ghitany(2005) discussed Marshall-Olkin extended Pareto dis-

tribution. Ristic et al.(2007) proposed Marshall-Olkin extended gamma distribution and minification process. Ghitany et al.(2007) discussed Marshall-Olkin extended Lomax using censored data. Srivastava et al.(2011) considered parameter estimation of Marshall-Olkin extended exponential distribution using MCMC method. Jose and Krishna(2011) proposed Marshall-Olkin extend Uniform distribution. Srivastava and Kumar (2011) estimated the two-parameter of the Marshall-Olkin extended Weibull using maximum likelihood estimate and Bayes estimate method. Jose et al.(2011) discussed Marshall-Olkin bivariate Weibull distribution and processes. Krishna et al.(2013a,b) introduced Marshall-Olkin Frèchet distribution and discussed its applications in reliability, sampling plan etc. Cordeiro and Lemonte (2013) studied some mathematical properties of Marshall-Olkin extended Weibull distribution. Jose et al.(2014) discuss on record values and reliability properties of Marshall-Olkin extended exponential distribution.

In this chapter we consider the Exponentiated Marshall-Olkin exponential and Weibull distribution. The generalization introduced by Jayakumar and Thomas (2008) is applied here to develop exponentiated Marshall-Olkin families of distributions. We introduce two new distributions namely, Exponentiated Marshall-Olkin exponential distribution and Exponentiated Marshall-Olkin Weibull distribution are considered. Various properties are studied including quantiles, order statistics, record values and Rényi entropy. Estimation of parameters is also considered. A real data set is analyzed as an application.

7.2 Exponentiated Marshall-Olkin Exponential Distribution

Here we consider the generalization of Marshall-Olkin family of distributions introduced by Jayakumar and Thomas (2008). Consider the survival function of exponential distribution $\bar{F}(x) = e^{-\lambda x}$; $\lambda > 0$ and introduce the Exponentiated Marshall-Olkin Exponential distribution (EMOE) with survival function given by

$$\bar{G}(x) = \left(\frac{\alpha e^{-\lambda x}}{1 - (1 - \alpha)e^{-\lambda x}} \right)^\gamma; \quad \alpha > 0, \gamma > 0, \lambda > 0. \quad (7.2.1)$$

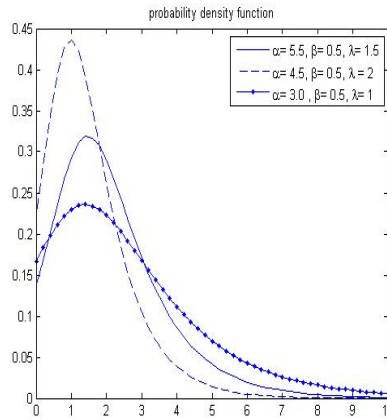


Figure 7.1: The p.d.f. of EMOE

Then the corresponding probability density function is given by

$$g(x; \alpha, \gamma, \lambda) = \frac{\gamma \lambda (\alpha e^{-\lambda x})^\gamma}{(1 - (1 - \alpha)e^{-\lambda x})^{\gamma+1}}; \alpha > 0, \gamma > 0, \lambda > 0.$$

When $\gamma = 1$, $g(x)$ becomes Marshall-Olkin Exponential distribution. When $\alpha = 1, \gamma = 1$ it becomes exponential distribution.

Figure 7.1 gives the probability density function of EMOE for various combinations of values of parameters.

The hazard rate function is given by

$$\begin{aligned} h(x; \alpha, \gamma) &= \frac{\gamma \lambda}{1 - (1 - \alpha)e^{-\lambda x}} \\ &= \frac{\gamma \lambda e^{\lambda x}}{e^{\lambda x} - (1 - \alpha)} \quad ; \alpha > 0, \gamma > 0, \lambda > 0 \end{aligned}$$

Thus the hazard rate function of this distribution is a multiple of γ as compared to the MOE distribution. Figure 7.2 shows the hazard rate function for various combination values of parameters.

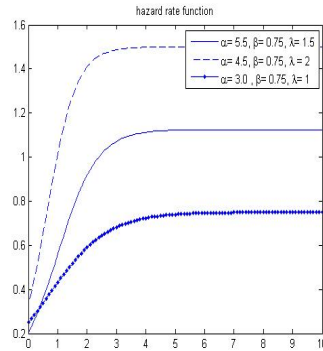


Figure 7.2: Hazard rate function of EMOE

7.2.1 Quantiles and Order statistics

The p^{th} quantile function of the distribution, obtained by taking the inverse of the distribution function $F(x_p) = p$, is given by

$$x = \frac{1}{\lambda} \log[\alpha(1-p)^{-\frac{1}{\gamma}} + (1-\alpha)]$$

Let X_1, X_2, \dots, X_n be a random sample taken from the EMOE distribution and $X_{1:n}, X_{2:n}, \dots, X_{n:n}$ be the corresponding order statistics. The survival function of EMOE distribution is given by (7.2.1). Then the c.d.f. of the first order statistic $X_{1:n}$ is given by

$$G_{1:n}(x) = 1 - \left(\frac{\alpha}{e^{\lambda x} - (1-\alpha)} \right)^{n\gamma}; \quad \alpha > 0, \gamma > 0, \lambda > 0.$$

The c.d.f. of the n^{th} order statistic $X_{n:n}$ is given by

$$\begin{aligned} G_{n:n}(x) &= \left[1 - \left(\frac{\alpha}{e^{\lambda x} - (1-\alpha)} \right)^{\gamma} \right]^n \\ &= \sum_{i=0}^n (-1)^i \binom{n}{i} \left[\frac{\alpha}{e^{\lambda x} - (1-\alpha)} \right]^{i\gamma}; \quad \alpha > 0, \gamma > 0, \lambda > 0. \end{aligned}$$

The probability density function of the i^{th} order statistics $X_{i:n}$ is given by

$$g_{i:n}(x) = \frac{n!}{(i-1)!(n-1)!} \frac{\alpha^\gamma \gamma \lambda e^{-\gamma \lambda x}}{[1 - (1-\alpha)e^{-\lambda x}]^{\gamma+1}} \left[1 - \frac{\alpha^\gamma e^{-\gamma \lambda x}}{[1 - (1-\alpha)e^{-\lambda x}]^\gamma} \right]^{i-1} \left[\frac{\alpha^\gamma e^{-\gamma \lambda x}}{[1 - (1-\alpha)e^{-\lambda x}]^\gamma} \right]^{n-i}$$

This can be written as a finite mixture of the probability density functions of EMOE distributed random variables since

$$g_{i:n}(x) = \frac{n!}{(i-1)!(n-1)!} \sum_{k=0}^{i-1} \frac{(-1)^k \binom{i-1}{k}}{n-i+k+1} g(x; \alpha, \gamma(n-i+k+1), \lambda)$$

7.2.2 Estimation of Parameters

Consider the estimation of unknown parameters by the method of maximum likelihood. For a given sample (x_1, x_2, \dots, x_n) , the log likelihood function is given by

$$\log L = n \log \gamma + n \log \lambda + n \gamma \log \alpha - \gamma \lambda \sum_{i=1}^n x_i - (\gamma + 1) \sum_{i=1}^n \log[1 - (1-\alpha)e^{-\lambda x_i}]$$

The partial derivatives of the log-likelihood function is given by

$$\frac{\partial \log L}{\partial \alpha} = \frac{n \gamma}{\alpha} - (\gamma + 1) \sum_{i=1}^n \frac{e^{-\lambda x_i}}{[1 - (1-\alpha)e^{-\lambda x_i}]}$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - \gamma \sum_{i=1}^n x_i - (\gamma + 1) \sum_{i=1}^n \frac{\lambda(1-\alpha)e^{-\lambda x_i}}{[1 - (1-\alpha)e^{-\lambda x_i}]}$$

$$\frac{\partial \log L}{\partial \gamma} = \frac{n}{\gamma} + n \log \alpha - \lambda \sum_{i=1}^n x_i - \sum_{i=1}^n [1 - (1 - \alpha)e^{-\lambda x_i}]$$

The maximum likelihood estimates can be obtained from solving $\frac{\partial \log L}{\partial \alpha} = 0$, $\frac{\partial \log L}{\partial \lambda} = 0$ and $\frac{\partial \log L}{\partial \gamma} = 0$. The solutions can be obtained by using nlm package in R software.

7.2.3 Rényi Entropy

Entropy is a measure of uncertainty regarding a random variable. Rényi entropy is a generalization of Shannon entropy. Rényi entropy of a random variable with probability density function is given by $I_R(\theta) = \frac{1}{1-\theta} \log \int_0^\infty g^\theta(x) dx$; $\theta > 0, \theta \neq 1$.

When $\theta = 1$ it reduces to Shannon entropy.

$$\int_0^\infty g^\theta(x) dx = \frac{(\gamma \lambda \alpha^\gamma)^\theta e^{-\theta \gamma \lambda x}}{[1 - (1 - \alpha)e^{-\lambda x}]^{\theta(\gamma+1)}}$$

If we put $u = e^{-\lambda x}$, we get

$$\begin{aligned} \int_0^\infty g^\theta(x) dx &= (\gamma \lambda \alpha^\gamma)^\theta \lambda^{-1} \int_0^1 \frac{u^{\gamma\theta-1}}{[1 - (1 - \alpha)u]^{\theta(\gamma+1)}} \\ &= (\gamma \lambda \alpha^\gamma)^\theta \lambda^{-1} {}_2F_1(\theta(\gamma + 1), \gamma\theta; 1 + \gamma\theta; 1 - \alpha) \end{aligned}$$

Therefore, the Rényi entropy is given by

$$\begin{aligned} I_R(\theta) &= \frac{1}{1-\theta} \log [(\gamma \lambda \alpha^\gamma)^\theta \lambda^{-1} {}_2F_1(\theta(\gamma + 1), \gamma\theta; 1 + \gamma\theta; 1 - \alpha)] \\ &= \frac{1}{1-\theta} [\theta \log(\gamma \lambda \alpha^\gamma) - \log \lambda + \log[{}_2F_1(\theta(\gamma + 1), \gamma\theta; 1 + \gamma\theta; 1 - \alpha)]] \end{aligned}$$

7.3 Exponentiated Marshall-Olkin Weibull Distribution

We consider the distribution function of Weibull distribution $F(x) = 1 - e^{-(\lambda x)^\beta}$; $\lambda > 0, \beta > 0, x > 0$ and introduce the Exponentiated Marshall-Olkin Weibull(EMOW) distribution given by the survival function

$$\bar{G}(x) = \left(\frac{\alpha e^{-(\lambda x)^\beta}}{1 - (1 - \alpha)e^{-(\lambda x)^\beta}} \right)^\gamma; \quad \alpha > 0, \beta > 0, \gamma > 0, \lambda > 0. \quad (7.3.1)$$

The corresponding probability density function is given by

$$g(x; \alpha, \beta, \gamma, \lambda) = \frac{\gamma \alpha^\gamma \beta \lambda^\beta x^{\beta-1} e^{-\gamma \lambda^\beta x^\beta}}{[1 - (1 - \alpha)e^{-\lambda^\beta x^\beta}]^{\gamma+1}}; \quad \alpha > 0, \beta > 0, \gamma > 0, \lambda > 0.$$

We denote it as EMOW($\alpha, \beta, \gamma, \lambda$). Then we have,

1. When $\gamma = 1$, the density becomes Marshall-Olkin Weibull distribution.
2. When $\alpha = 1, \gamma = 1$, it becomes the Weibull distribution.
3. When $\beta = 1, \gamma = 1$, it becomes Marshall-Olkin Exponential distribution and
4. When $\alpha = 1, \beta = 1, \gamma = 1$, it becomes the Exponential distribution.

The hazard rate function of the EMOW distribution is given by

$$h(x; \alpha, \beta, \gamma, \lambda) = \frac{\gamma \beta \lambda^\beta x^{\beta-1} e^{\lambda^\beta x^\beta}}{e^{\lambda^\beta x^\beta} - (1 - \alpha)}$$

Figure 7.1 and Figure 7.2 shows the probability density function and hazard rate function for different combination values of parameters.

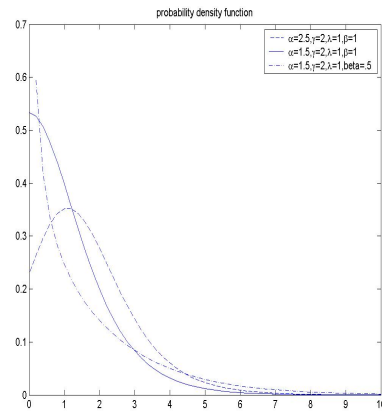


Figure 7.3: The p.d.f. of EMOW

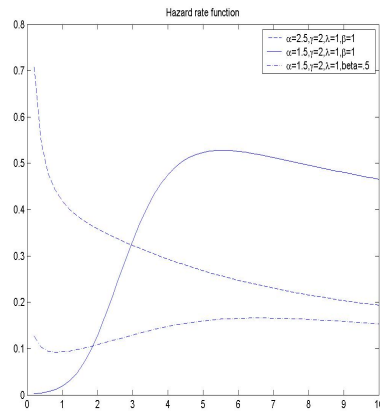


Figure 7.4: Hazard rate function for EMOW

7.3.1 Quantiles and Order statistics

The p^{th} quantile function of the distribution, the inverse of the distribution function $F(x_p) = p$, is given by

$$x = \frac{1}{\lambda} \log[\alpha(1-p)^{\frac{-1}{\gamma}} + (1-\alpha)]^{\frac{1}{\gamma}}$$

Let X_1, X_2, \dots, X_n be a random sample taken from the EMOW distribution and $X_{1:n}, X_{2:n}, \dots, X_{n:n}$ be the corresponding order statistics. The survival function of EMOW distribution is given by (7.3.1). Then the c.d.f. of the first order statistic $X_{1:n}$ is given by

$$G_{1:n}(x) = 1 - \left(\frac{\alpha}{e^{(\lambda x)^\beta} - (1-\alpha)} \right)^{n\gamma}; \quad \alpha > 0, \gamma > 0, \lambda > 0.$$

The c.d.f. of the n^{th} order statistic $X_{n:n}$ is given by

$$\begin{aligned} G_{n:n}(x) &= \left[1 - \left(\frac{\alpha}{e^{(\lambda x)^\beta} - (1-\alpha)} \right)^\gamma \right]^n \\ &= \sum_{i=0}^n (-1)^i \binom{n}{i} \left[\frac{\alpha}{e^{(\lambda x)^\beta} - (1-\alpha)} \right]^{i\gamma}; \quad \alpha > 0, \gamma > 0, \lambda > 0. \end{aligned}$$

The probability density function of the i^{th} order statistics $X_{i:n}$ is given by

$$\begin{aligned} g_{i:n}(x; \alpha, \beta, \gamma, \lambda) &= \frac{n!}{(i-1)!(n-1)!} \frac{\gamma \alpha^\gamma \beta \lambda^\beta x^{\beta-1} e^{-\gamma(\lambda x)^\beta}}{[1 - (1-\alpha)e^{-(\lambda x)^\beta}]^{\gamma+1}} \\ &\quad \left[1 - \frac{(\alpha e^{-(\lambda x)^\beta})^\gamma}{[1 - (1-\alpha)e^{-(\lambda x)^\beta}]^\gamma} \right]^{i-1} \left[\frac{(\alpha e^{-(\lambda x)^\beta})^\gamma}{[1 - (1-\alpha)e^{-(\lambda x)^\beta}]^\gamma} \right]^{n-i} \end{aligned}$$

This can be written as a finite mixture of the probability density functions of EMOW

distributed random variables since

$$g_{i:n}(x; \alpha, \beta, \gamma, \lambda) = \frac{n!}{(i-1)!(n-1)!} \sum_{k=0}^{i-1} \frac{(-1)^k \binom{i-1}{k}}{n-i+k+1} g(x; \alpha, \beta, \gamma(n-i+k+1), \lambda)$$

7.3.2 Record values

Record values are used in reliability theory. Chandler (1952) introduced a statistical study of record values as a model for successive extremes in a sequence of i.i.d random variables. The theory of record values and its distributional properties are studied by Ahsanullah (1995,1997), Raqab (2001), Balakrishnan and Ahsanullah (1994), Saran and Singh (2008) and Jose et al.(2014) discusses on record values from Marshall-Olkin Extended Exponential distribution.

Here we consider the record statistics of Exponentiated Marshall-Olkin Weibull distribution with $\gamma = \beta = 1$ with the pdf is given by

$$g(x) = \frac{\alpha\lambda}{[e^{\lambda x} - (1-\alpha)]^2}; 0 < x < \infty \quad (7.3.2)$$

Using (1.3.1), we get the pdf of n^{th} record of EMOW (α, λ) given by

$$g_{R_n}(x) = \frac{[\log[e^{\lambda x} - (1-\alpha)] - \log \alpha]^{n-1}}{(n-1)!} \frac{\alpha\lambda}{[e^{\lambda x} - (1-\alpha)]^2}$$

Using (1.3.2) , we get the joint pdf of m^{th} and n^{th} record statistics of EMOW(α, λ)

$$g_{R_m, R_n}(x, y) = \frac{[\log[e^{\lambda x} - (1-\alpha)] - \log \alpha]^{m-1}}{(m-1)!} \frac{[\log[e^{\lambda y} - (1-\alpha)] - \log[e^{\lambda x} - (1-\alpha)]]^{n-m-1}}{(n-m-1)!} \frac{\lambda^2 e^{-\lambda y}}{[1 - (1-\alpha)e^{-\lambda x}][1 - (1-\alpha)e^{-\lambda y}]^2}$$

7.3.3 Estimation of Parameters

Consider the estimation of unknown parameters by the method of maximum likelihood. For a given sample (x_1, x_2, \dots, x_n) , the log likelihood function is given by

$$\begin{aligned} \log L = & n \log \gamma + n\gamma \log \alpha + n \log \beta + n\beta \log \lambda + (\beta - 1) \sum_{i=0}^n x_i \\ & - \gamma \lambda^\beta \sum_{i=0}^n x_i^\beta - (\gamma + 1) \sum_{i=0}^n \log[1 - (1 - \alpha)e^{-(\lambda x_i)^\beta}] \end{aligned}$$

The partial derivatives of the log-likelihood function is given by

$$\frac{\partial \log L}{\partial \alpha} = \frac{n\gamma}{\alpha} - (\gamma + 1) \sum_{i=1}^n \frac{e^{-(\lambda x_i)^\beta}}{[1 - (1 - \alpha)e^{-(\lambda x_i)^\beta}]}$$

$$\begin{aligned} \frac{\partial \log L}{\partial \beta} = & \frac{n}{\beta} + n \log \lambda + \sum_{i=0}^n x_i - \gamma \lambda^\beta \log \lambda \sum_{i=0}^n x_i^\beta - \gamma \lambda^\beta \sum_{i=0}^n x_i^\beta \log x_i \\ & - (\gamma + 1) \sum_{i=0}^n \frac{(1 - \alpha)e^{-(\lambda x_i)^\beta} [(\lambda x_i)^\beta (\log \lambda + \log x_i)]}{[1 - (1 - \alpha)e^{-(\lambda x_i)^\beta}]} \end{aligned}$$

$$\frac{\partial \log L}{\partial \gamma} = \frac{n}{\gamma} + n \log \alpha - \lambda^\beta \sum_{i=1}^n x_i^\beta - \sum_{i=1}^n \log[1 - (1 - \alpha)e^{-(\lambda x_i)^\beta}]$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n\beta}{\lambda} - \gamma \beta \lambda^{\beta-1} \sum_{i=0}^n x_i^\beta - (\gamma + 1) \sum_{i=0}^n \frac{(1 - \alpha)\beta \lambda^{\beta-1} x_i^\beta e^{-(\lambda x_i)^\beta}}{[1 - (1 - \alpha)e^{-(\lambda x_i)^\beta}]}$$

The maximum likelihood estimates can be obtained from solving $\frac{\partial \log L}{\partial \alpha} = 0$, $\frac{\partial \log L}{\partial \lambda} = 0$

and $\frac{\partial \log L}{\partial \gamma} = 0$. The solutions can be obtained by using nlm package in R software.

7.4 Data Analysis

Here we consider the applications of the models to two data sets. The first data set is used to compare Exponentiated Marshall-Olkin Exponential distribution with Marshall-Olkin Exponential and Exponential distribution and the second data is used to compare Exponentiated Marshall-Olkin Weibull distribution with Marshall-Olkin Weibull and Weibull distribution. To compare the goodness of fit, we use the information criteria $AIC = -2 \log L + 2k$, $BIC = -2 \log L + k \log n$ and the Kolmogorov-Smirnov statistic, where k is the no. of parameters and n is the sample size.

Data set 1: This data is taken from Lawless (2003). The data set gives the number of cycles to failure for 25 100-cm specimens of yarn, tested at a particular strain level and they are 15, 20, 38, 42, 61, 76, 86, 98, 121, 146, 149, 157, 175, 176, 180, 180, 198, 220, 224, 251, 264, 282, 321, 325, 653. The maximum likelihood estimates, AIC and BIC values, and K-S statistic are given in Table 7.1. From Table 7.1, it is seen that the smallest AIC and BIC values are obtained for EMOE distribution. Hence we conclude that EMOE distribution is a better model for the data set.

Data set 2: This data is taken from Nadarajah (2008). The following data are the daily ozone measurements in New York, May-September 1973: 41, 36, 12, 18, 28, 23, 19, 8, 7, 16, 11, 14, 18, 14, 34, 6, 30, 11, 1, 11, 4, 32, 23, 45, 115, 37, 29, 71, 39, 23, 21, 37, 20, 12, 13, 135, 49, 32, 64, 40, 77, 97, 97, 85, 10, 27, 7, 48, 35, 61, 79, 63, 16, 80, 108, 20, 52, 82, 50, 64, 59, 39, 9, 16, 78, 35, 66, 122, 89, 110, 44, 28, 65, 22, 59, 23, 31, 44, 21, 9, 45, 168, 73, 76, 118, 84, 85, 96, 78, 73, 91, 47, 32, 20, 23, 21, 24, 44, 21, 28, 9, 13, 46, 18, 13, 24, 16, 13, 23, 36, 7, 14, 30, 14, 18, 20. The maximum likelihood estimates, AIC and BIC values, and K-S statistic are given in Table 7.2. From Table 7.2, it is seen that the smallest AIC and BIC values are obtained for EMOW distribution. Hence we conclude that EMOW distribution is a better model for the data set.

Table 7.1: Estimates, AIC, BIC, and K-S statistic for the data set 1

Model	Parameter	Estimates	AIC	BIC	K-S statistic
EMOE	α	4.9830	310.3464	308.54022	0.1099
	γ	0.6717			
	λ	0.0147			
MOE	α	1.0512	312.9172	311.71308	0.1862
	λ	0.0058			
Exponential	λ	0.0056	311.179	310.57694	0.1985

Table 7.2: Estimates, AIC , BIC and K-S statistic for the data set 2

Model	Parameter	Estimates	AIC	BIC	K-S statistic
EMOW	α	24.045	948.3768	948.7644	0.0899
	β	0.4022			
	γ	82.067			
	λ	0.0074			
MOW	α	0.7028	1205.4056	1205.6963	0.5132
	β	0.7180			
	λ	0.0927			
Weibull	β	1.3402	1089.2206	1089.4144	0.2863
	λ	0.0217			

7.5 Conclusion

In this chapter we consider a generalization of Marshall-Olkin family of distributions. First we consider EMOE distribution and its properties. Rényi entropy is discussed. Also Exponentiated Marshall-Olkin Weibull Distribution and its properties, record values etc. are considered. We analyze two real data sets. First data is used to compare EMOE with MOE and exponential, and the other to compare EMOW with MOW and Weibull distribution. We conclude from the two data sets that, EMOE and EMOW are better fits as compared to others. The results are given in Table 7.1 and Table 7.2.

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CHAPTER 8

Negative Binomial Extreme Stable Marshall-Olkin Extended Lindley Distribution and its Applications

8.1 Introduction

Lindley (1958) introduced a distribution as life time model and suggested its applications for studying stress-strength model in reliability. The probability density function of Lindley random variable X , with scale parameter λ is given by

$$f(x, \lambda) = \frac{\lambda^2}{1 + \lambda} (1 + x) e^{-\lambda x}; x > 0, \lambda > 0 \quad (8.1.1)$$

The corresponding cumulative distribution function (c.d.f.) is given by

$$F(x, \lambda) = 1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} e^{-\lambda x} \quad (8.1.2)$$

It can be seen that this distribution is a mixture of exponential(λ) and gamma(2, λ) distributions. Lindley distribution has drawn much attention in the statistical literature over the great popularity of the well-known exponential distribution. Sankaran (1970) introduced the discrete Poisson-Lindley distribution by combining the Poisson and Lindley distributions. Ghitany, Atieh, Nadarajah(2008) investigated most of the statistical properties of the Lindley distribution. Mahmoudi and Zakerzadeh (2010) proposed an extended version of the compound Poisson distribution which was obtained by compounding the Poisson distribution with the generalized Lindley distribution.

Alice and Jose(2005) discuss Marshall-Olkin semi- Weibull minification processes and Jose et al.(2010) also proposed Marshall-Olkin q-Weibull distribution and maxmin processes. Jayakumar and Thomas(2008) developed a generalization of Marshall-Olkin distribution and discuss its application to Burr Type XII distribution. Jose et al.(2011) discussed Marshall-Olkin bivariate Weibull distributions and applications. Louzada, Roman and Cancho(2011) proposed the complimentary exponential geometric distribution by combining the geometric and exponential distributions. Bakouch et al.(2012) introduced a new extension of the Lindley distribution, called extended Lindley (EL)distribution, which offers a more flexible model for lifetime data. Mazucheli and Achcar(2011) discussed the applications of Lindley distribution to competing risks lifetime data. Ghitany et al.(2011) developed a two-parameter weighted Lindley distribution and discussed its applications to survival data. Ghitany et al.(2012) also discusses Marshall-Olkin Extended Lindley distribution and its application. Zakerzadah and Dolati(2010) obtained a generalized Lindley distribution and discussed its various properties and applications. Al-Mutairi et al.(2013) developed inferences on stress-strength reliability from Lindley distributions.

The Lindley distribution is important for studying stress-strength reliability modeling. The stress-strength reliability has been originally proposed by Birnbaum (1956). Birnbaum and McCarty (1958) and Govindarajulu (1968) have discussed the procedure for obtaining the distribution free confidence interval for stress strength reliability and Cheng and Chao (1984) have compared the performances of different methods of constructing the confidence interval of stress strength reliability and proposed a new method for obtaining the s-confidence intervals for the reliability in the stress-strength model. Shanker and Mishra (2013) introduced a two parameter quasi Lindley distribution as a particular case of Lindley distribution. A new generalization of Lindley distribution called Transmuted Quasi Lindley distribution was introduced by Elbatal and Elgarhy (2013).

In this chapter we introduce Negative Binomial Extreme Stable Marshall-Olkin Extended Lindley Distribution and its Applications. First we consider the properties of Extended Lindley distribution. Negative binomial extreme stable Marshall-Olkin Extended Lindley Distribution and its properties are discussed. The quantiles and order statistics are obtained. Record values associated with the new family is also considered. The maximum likelihood estimates of the distribution is obtained by using R programme and is applied to a real data set.

8.2 Extended Lindley Distribution

Consider a particular exponentiation of (8.1.2) to extend the Lindley distribution, for which the distribution function is given by,

$$F(x) = 1 - \left(\frac{1 + \lambda + \lambda x}{1 + \lambda} \right)^\theta e^{-(\lambda x)^\beta} \quad (8.2.1)$$

where $\theta \in \mathbb{R}^- \cup \{0, 1\}$, $\lambda > 0$, and $\beta \geq 0$.

The extension of the Lindley distribution shall be denoted by extended Lindley (EL)distribution.

EL distribution has several particular cases. For $\theta = 1$ and $\beta = 1$ the EL distribution is reduced to Lindley distribution and for $\theta = 0$ it reduces to the Weibull distribution. Also for $\beta = 0$ and $\theta \in \mathbb{R}^-$, the EL distribution reduces to Pareto distribution given by $F(x) = 1 - \left(\frac{c}{1+c}\right)^\delta$, where $x > 0$, $c = 1 + \frac{1}{\lambda}$ and $\delta > 0$. (8.2.1) represents the product of the survival functions $(1 - F(x))$ of the Lomax and Weibull distribution respectively, for any β and $\alpha \in \mathbb{R}^-$ (Murthy,Swartz,and Yuen,1973). Ghitany et al.(2007) discussed Marshall-Olkin extended Lomax distributions and its application to censored data. Extended Lindley distribution can be seen as a mixture of Lomax and Weibull distribution respectively.

The probability density function of extended Lindley distribution is given by

$$f(x) = \frac{\lambda(1 + \lambda + \lambda x)^{\theta-1}}{(1 + \lambda)^\theta} (\beta(1 + \lambda + \lambda x)(\lambda x)^{\beta-1} - \theta) e^{-(\lambda x)^\beta}; x > 0 \quad (8.2.2)$$

and the corresponding survival function and hazard rate function are given by

$$\bar{F}(x) = \left(\frac{1 + \lambda + \lambda x}{1 + \lambda}\right)^\theta e^{-(\lambda x)^\beta}; x > 0, \quad (8.2.3)$$

and

$$r(x) = \frac{\beta(1 + \lambda + \lambda x)\lambda^\beta x^{\beta-1} - \lambda\theta}{1 + \lambda + \lambda x} \quad (8.2.4)$$

The first derivative of $r(x)$ is

$$r'(x) = \beta(\beta - 1)\lambda^\beta x^{\beta-2} + \frac{\lambda^2\theta}{(1 + \lambda + \lambda x)^2} \quad (8.2.5)$$

It is obvious that $r'(x) \leq 0$, for $\beta \leq 1$ and $\theta \leq 0$. The function $r(x)$ is increasing for $\theta > k$ and decreasing for $\theta < k$, where $k = -\beta(\beta - 1)(\lambda x)^{\beta-2}(1 + \lambda + \lambda x)^2$. For $\beta > 1$, $r(0) = f(0) = \frac{\lambda\theta}{1+\lambda}$. Therefore at the origin $r(x)$ varies continuously with the

parameters. This is in contrast with the Weibull and gamma families, where $r(0) = 0$ or $r(0) = \infty$ for both families and hence $r(0)$ is discontinuous in the parameters of such families. For $\beta = 1$, $\lim_{x \rightarrow \infty} r(x) = \lambda$, the function $r(x)$ is bounded above by λ and continuous in the parameters of the EL distribution.

The p^{th} quantile x_p of the EL distribution, the inverse of the distribution function $F(x_p) = p$ is given by

$$x_p = \left(\frac{\theta}{\lambda^\beta} \ln \left(\frac{1 + \lambda + \lambda x_p}{(1 + \lambda)(1 - p)^{\frac{1}{\theta}}} \right) \right)^{\frac{1}{\beta}}$$

8.3 Negative Binomial Extreme Stable Marshall- Olkin Extended Lindley Distribution

We consider the survival function of extended Lindley distribution $\bar{F}(x) = \left(\frac{1+\lambda+\lambda x}{1+\lambda} \right)^\theta e^{-(\lambda x)^\beta}$, $x > 0$. From negative binomial extreme stable family, we introduce Negative Binomial Extreme Stable Marshall-Olkin Extended Lindley distribution. It is denoted by NBESMOEL distribution. The survival function of negative binomial extreme stable Marshall-Olkin extended Lindley distribution is given by

$$\bar{G}(x) = \frac{\alpha^\gamma}{1 - \alpha^\gamma} \left[\left(1 - (1 - \alpha) \left(\frac{1 + \lambda + \lambda x}{1 + \lambda} \right)^\theta e^{-(\lambda x)^\beta} \right)^{-\gamma} - 1 \right] \quad (8.3.1)$$

Then the corresponding probability density function is given by

$$g(x) = \frac{(1 - \alpha)\gamma\alpha^\gamma\lambda\frac{(1+\lambda+\lambda x)^{\theta-1}}{(1+\lambda)^\theta}[\beta(1 + \lambda + \lambda x)(\lambda x)^{\beta-1} - \theta]e^{-(\lambda x)^\beta}}{(1 - \alpha^\gamma) \left\{ 1 - (1 - \alpha) \left(\frac{1+\lambda+\lambda x}{1+\lambda} \right)^\theta e^{-(\lambda x)^\beta} \right\}^{\gamma+1}} \quad (8.3.2)$$

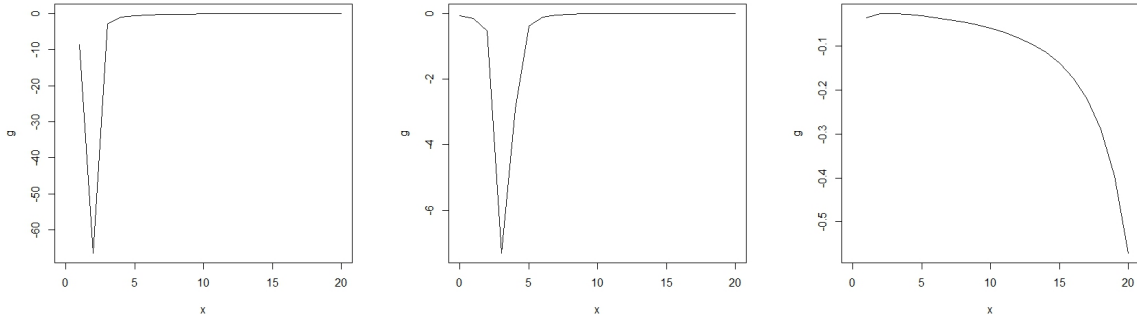


Figure 8.1: The p.d.f. of NBESMOEL for various values of parameters.

for $\alpha < 1, \beta, \gamma, \lambda, \theta > 0$ and the cumulative distribution function is given by

$$G(x) = 1 - \frac{\alpha^\gamma}{1 - \alpha^\gamma} \left[\left(1 - (1 - \alpha) \left(\frac{1 + \lambda + \lambda x}{1 + \lambda} \right)^\theta e^{-(\lambda x)^\beta} \right)^{-\gamma} - 1 \right] \quad (8.3.3)$$

Figure 8.1, figure 8.2 and figure 8.3 shows the pdf of NBESMOEL for different combinations of parameter values.

The hazard rate function is given by

$$h(x) = \frac{(1 - \alpha)\gamma\bar{F}(x)r_F(x)}{(1 - (1 - \alpha)\bar{F}(x))[1 - (1 - (1 - \alpha)\bar{F}(x))^\gamma]}$$

$$r(x) = \frac{(1 - \alpha)\gamma\lambda(1 + \lambda + \lambda x)^{\theta-1}}{\left((1 + \lambda)^\theta - (1 - \alpha)(1 + \lambda + \lambda x)^\theta e^{-(\lambda x)^\beta} \right)} \frac{[\beta(1 + \lambda + \lambda x)(\lambda x)^{\beta-1} - \theta]e^{-(\lambda x)^\beta}}{\left[1 - \left\{ 1 - (1 - \alpha) \left(\frac{1 + \lambda + \lambda x}{1 + \lambda} \right)^\theta e^{-(\lambda x)^\beta} \right\} \right]}$$

Figure 8.4, figure 8.5 and figure 8.6 show the hazard rate function of NBESMOEL for

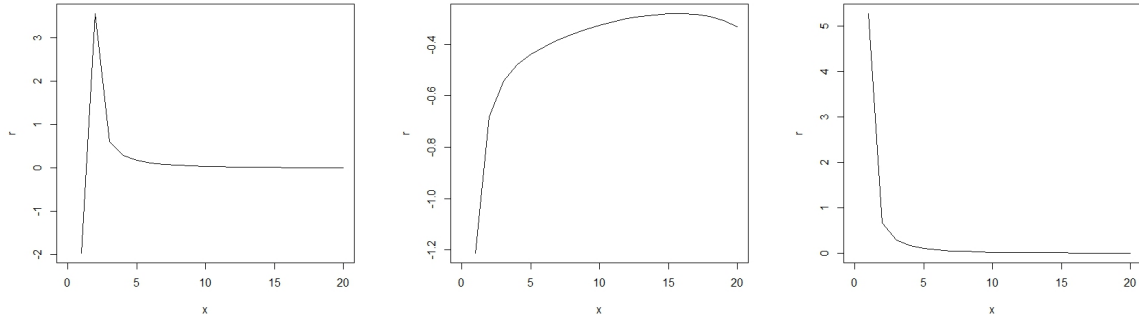


Figure 8.2: Hazard rate function of NBESMOEL for various values of parameters.

different combinations of parameter values.

8.4 Quantiles and Order statistics

The p^{th} quantile function of the distribution is given by

$$x = \left\{ \frac{1}{\lambda^\beta} \ln \left[\left(1 - \frac{\alpha}{((1-p)(1-\alpha^\gamma) + \alpha^\gamma)^{\frac{1}{\gamma}}} \right)^{-1} \left(\frac{1+\lambda+\lambda x_p}{1+\lambda} \right)^\theta (1-\alpha) \right] \right\}^{\frac{1}{\beta}}$$

Let X_1, X_2, \dots, X_n be a random sample taken from the NBESMOEL distribution and $X_{1:n}, X_{2:n}, \dots, X_{n:n}$ be the corresponding order statistics. The survival function of NBESMOEL distribution is given by (8.3.1). Then the c.d.f of the minimum order statistic $X_{1:n}$ is given by

$$\begin{aligned} G_{1:n}(x) &= 1 - (\bar{G}(x))^n \\ &= 1 - \left\{ \frac{\alpha^\gamma}{1-\alpha^\gamma} \left(\left[1 - (1-\alpha) \left(\frac{1+\lambda+\lambda x}{1+\lambda} \right)^\theta e^{-(\lambda x)^\beta} \right]^{-\gamma} - 1 \right) \right\}^n \end{aligned}$$

The c.d.f of the maximum order statistic $X_{n:n}$ is given by

$$\begin{aligned} G_{n:n}(x) &= [1 - \bar{G}(x)]^n \\ &= \left[1 - \left\{ \frac{\alpha^\gamma}{1 - \alpha^\gamma} \left(\left[1 - (1 - \alpha) \left(\frac{1 + \lambda + \lambda x}{1 + \lambda} \right)^\theta e^{-(\lambda x)^\beta} \right]^{-\gamma} - 1 \right) \right\} \right]^n \end{aligned}$$

The probability density function $g_{i:n}(x)$ of the i^{th} order statistics $X_{i:n}$ is given by

$$\begin{aligned} g_{i:n}(x) &= \frac{n!(1 - \alpha)\gamma\alpha^\gamma}{(i - 1)!(n - i)!(1 - \alpha^\gamma) \left\{ 1 - (1 - \alpha) \left(\frac{1 + \lambda + \lambda x}{1 + \lambda} \right)^\theta e^{-(\lambda x)^\beta} \right\}^{\gamma+1}} \\ &\quad \left\{ 1 - \frac{\alpha^\gamma}{1 - \alpha^\gamma} \left(\left[1 - (1 - \alpha) \left(\frac{1 + \lambda + \lambda x}{1 + \lambda} \right)^\theta e^{-(\lambda x)^\beta} \right]^{-\gamma} - 1 \right) \right\}^{i-1} \\ &\quad \left(\frac{\alpha^\gamma}{1 - \alpha^\gamma} \right)^{n-i} \left\{ \left(\left[1 - (1 - \alpha) \left(\frac{1 + \lambda + \lambda x}{1 + \lambda} \right)^\theta e^{-(\lambda x)^\beta} \right]^{-\gamma} - 1 \right) \right\}^{n-i} \\ &\quad \frac{(1 + \lambda + \lambda x)^{\theta-1}}{(1 + \lambda)^\theta} [\beta(1 + \lambda + \lambda x)(\lambda x)^{\beta-1} - \theta] e^{-(\lambda x)^\beta} \end{aligned}$$

8.5 Record values

In this section we consider the record statistics of Negative binomial extreme stable Marshall-Olkin extended Lindley distribution with $\gamma = \theta = \beta = 1$ with the pdf given by

$$g(x) = \frac{\alpha\lambda(1 + \lambda)(\lambda + \lambda x)e^{-(\lambda x)}}{[(1 + \lambda) - (1 - \alpha)(1 + \lambda + \lambda x)e^{-(\lambda x)}]^2}; 0 < x < \infty \quad (8.5.1)$$

Using (1.3.1) and (1.3.2), we get the pdf and joint pdf of NBESMOEL(α, λ) as

$$g_{R_n}(x) = \frac{\alpha\lambda(1+\lambda)(\lambda+\lambda x)e^{-(\lambda x)}}{[(1+\lambda) - (1-\alpha)(1+\lambda+\lambda x)e^{-(\lambda x)}]^2} \times \left[-\log \left\{ \frac{\alpha(1+\lambda+\lambda x)e^{-(\lambda x)}}{(1+\lambda) - (1-\alpha)(1+\lambda+\lambda x)e^{-(\lambda x)}} \right\} \right]^{n-1} \quad (8.5.2)$$

$$g_{R_m, R_n}(x, y) = \frac{1}{(n-1)!(n-m-1)!} \left[-\log \left\{ \frac{\alpha(1+\lambda+\lambda x)e^{-(\lambda x)}}{(1+\lambda) - (1-\alpha)(1+\lambda+\lambda x)e^{-(\lambda x)}} \right\} \right]^{n-1} \left[\log \frac{(1+\lambda+\lambda x)e^{-(\lambda x)}}{(1+\lambda+\lambda y)e^{-(\lambda y)}} \frac{[(1+\lambda) - (1-\alpha)(1+\lambda+\lambda y)e^{-(\lambda y)}]}{[(1+\lambda) - (1-\alpha)(1+\lambda+\lambda x)e^{-(\lambda x)}]} \right]^{n-m-1} \frac{(1+\lambda)^2\alpha\lambda^2(\lambda+\lambda x)(\lambda+\lambda y)e^{-(\lambda y)}}{[(1+\lambda) - (1-\alpha)(1+\lambda+\lambda x)e^{-(\lambda x)}]} \frac{1}{[(1+\lambda) - (1-\alpha)(1+\lambda+\lambda y)e^{-(\lambda y)}] (1+\lambda+\lambda x)}$$

8.6 Estimation of Parameters

In this section we consider maximum likelihood estimation with respect to a given sample of size x_1, x_2, \dots, x_n , then the log likelihood function is given by

$$\begin{aligned} \log L(\alpha, \beta, \gamma, \lambda, \theta) &= n \log(1-\alpha) + n \log \gamma + n \gamma \log \alpha + n \log \lambda + (\theta-1) \log(1+\lambda+\lambda x_i) \\ &+ n \log \beta + \sum_{i=1}^n \log(1+\lambda+\lambda x_i) + (\beta-1) \sum_{i=1}^n \log(\lambda x_i) - n \log \theta \\ &+ \lambda^\beta \sum_{i=1}^n x_i^\beta - n \log(1-\alpha^\gamma) - n \theta \log(1+\lambda) \\ &- (\gamma+1) \sum_{i=1}^n \log \left[1 - (1-\alpha) \left(\frac{1+\lambda+\lambda x_i}{1+\lambda} \right)^\theta e^{-(\lambda x_i)^\beta} \right] \end{aligned}$$

The partial derivatives of the log likelihood functions with respect to the parameters

are

$$\frac{\partial \log L}{\partial \alpha} = \frac{-n}{1-\alpha} + \frac{n\gamma}{\alpha} + \frac{n\gamma\alpha^{\gamma-1}}{1-\alpha^\gamma} + (\gamma+1) \sum_{i=1}^n \frac{\left(\frac{1+\lambda+\lambda x_i}{1+\lambda}\right)^\theta e^{-(\lambda x_i)^\beta}}{1 - (1-\alpha) \left(\frac{1+\lambda+\lambda x_i}{1+\lambda}\right)^\theta e^{-(\lambda x_i)^\beta}}$$

$$\begin{aligned} \frac{\partial \log L}{\partial \beta} &= \frac{n}{\beta} + \sum_{i=1}^n \log(\lambda x_i) + \lambda^\beta \log \lambda + x_i^\beta \log x_i \\ &+ (\gamma+1)(1-\alpha)\beta\lambda^\beta \sum_{i=1}^n \frac{\left(\frac{1+\lambda+\lambda x_i}{1+\lambda}\right)^\theta e^{-(\lambda x_i)^\beta} x_i^{\beta-1}}{1 - (1-\alpha) \left(\frac{1+\lambda+\lambda x_i}{1+\lambda}\right)^\theta e^{-(\lambda x_i)^\beta}} \end{aligned}$$

$$\frac{\partial \log L}{\partial \gamma} = \frac{n}{\gamma} + n \log \alpha + \frac{n\alpha^\gamma \log \alpha}{1-\alpha^\gamma} - \sum_{i=1}^n \log \left[1 - (1-\alpha) \left(\frac{1+\lambda+\lambda x_i}{1+\lambda}\right)^\theta e^{-(\lambda x_i)^\beta} \right]$$

$$\begin{aligned} \frac{\partial \log L}{\partial \lambda} &= \frac{n}{\lambda} + \frac{(\theta-1)(1+x_i)}{1+\lambda+\lambda x_i} + \sum_{i=1}^n \frac{1+x_i}{1+\lambda+\lambda x_i} + (\beta-1)n \sum_{i=1}^n \frac{x_i}{\lambda x_i} \\ &+ \beta\lambda^{\beta-1} \sum_{i=1}^n x_i^\beta - \frac{n\theta}{1+\lambda} + (\gamma+1) \sum_{i=1}^n \frac{1}{1 - (1-\alpha) \left(\frac{1+\lambda+\lambda x_i}{1+\lambda}\right)^\theta e^{-(\lambda x_i)^\beta}} \\ &\left[(1-\alpha) \left\{ - \left(\frac{1+\lambda+\lambda x_i}{1+\lambda}\right)^\theta e^{-(\lambda x_i)^\beta} \beta x_i^\beta \lambda^{\beta-1} + \theta \left[\frac{1+\lambda+\lambda x_i}{1+\lambda}\right]^{\theta-1} \frac{x_i e^{-(\lambda x_i)^\beta}}{(1+\lambda)^2} \right\} \right] \end{aligned}$$

$$\begin{aligned} \frac{\partial \log L}{\partial \theta} &= \log(1+\lambda+\lambda x_i) - \frac{n}{\theta} - n \log(1+\lambda) \\ &- (\gamma+1)(1-\alpha) \sum_{i=1}^n \frac{\left(\frac{1+\lambda+\lambda x_i}{1+\lambda}\right)^\theta \log \left(\frac{1+\lambda+\lambda x_i}{1+\lambda}\right) e^{-(\lambda x_i)^\beta}}{1 - (1-\alpha) \left(\frac{1+\lambda+\lambda x_i}{1+\lambda}\right)^\theta e^{-(\lambda x_i)^\beta}} \end{aligned}$$

The maximum likelihood estimates can be obtained by solving the equations $\frac{\partial \log L}{\partial \alpha} =$

Table 8.1: Estimates, -log likelihood and K-S statistic for the data

Model	Parameter	Estimates	-log likelihood	k-s statistic
NBESMOEL	α	2.8352	128.32	0.7624
	β	0.1736		
	γ	0.8722		
	λ	1.5481		
	θ	0.2915		
EL	β	0.2044	131.14	0.8531
	λ	1.3956		
	θ	0.0119		

$0, \frac{\partial \log L}{\partial \beta} = 0, \frac{\partial \log L}{\partial \gamma} = 0, \frac{\partial \log L}{\partial \lambda} = 0, \frac{\partial \log L}{\partial \theta} = 0$. The equations can be solved using nlm package in R software.

8.7 Data Analysis

In this section we analyze a data set and compare Negative Binomial Extreme Stable Marshall-Olkin Extended Lindley distribution with Extended Lindley distribution. We consider data from Linhart and Zucchini (1986). The following data are failure times of the air conditioning system of an airplane: 23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95.

Extended Lindley distribution with parameters β, λ and θ and Negative Binomial Extreme Stable Marshall-Olkin Extended Lindley distribution with parameters $\alpha, \beta, \gamma, \lambda$ and θ are fitted to the data. The results are presented in Table 8.1. The QQ plots for the two distributions are shown in Figure 8.7. From Table 8.1, it is seen that the K-S statistic for the NBESMOEL distribution is 0.7624 which is less than that for the EL distribution. Similar is the case with log likelihood values. The Q-Q plot also confirm that the new distribution fits well than the original EL distribution. Hence we conclude that NBESMOEL distribution is a better model for the data set.

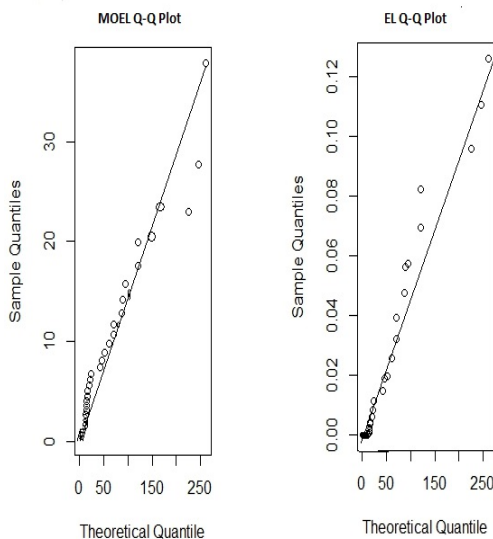


Figure 8.3: QQ plot for NBESMOEL and EL Distribution

8.8 Conclusion

In this chapter we proposed a new distribution namely, Negative Binomial Extreme Stable Marshall-Olkin Extended Lindley distribution. Its properties are obtained. Record values and estimation of parameters are also discussed. We analyze a real data set and compare the goodness of fit to NBESMOEL with EL distribution. We conclude that NBESMOEL distribution is a better fit. The results are given in Table 8.1.

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CHAPTER 9

Reliability Test Plan for Negative Binomial Extreme Stable Marshall- Olkin Pareto Distribution

9.1 Introduction

A wide variety of socioeconomic data have distributions which are heavy tailed and reasonably fitted by Pareto distribution. Pareto distribution is used to analyze the stock price fluctuations, insurance risks, business failures etc. Davis and Feldstin (1979) introduced Pareto distribution as a model for survival data. Pareto distribution can be considered as a lifetime distribution and is shown to be a decreasing failure rate model. Addel-Ghaly et al.[1998] discussed estimation of the parameters of Pareto distribution and reliability function using accelerated life testing with censoring, Kulldorff and Vannman[1973]

studied estimation of the location and scale parameters of a Pareto distribution by linear functions of order statistics, Vannman[1976] introduced estimators based on order statistics from a Pareto distribution. Alice and Jose (2003) introduced Marshall-Olkin extended semi Pareto model for Pareto type III and established its geometric extreme stability. Ghitany(2005) introduced Marshall-Olkin extended Pareto distribution.

Acceptance sampling plan is an inspection procedure used to determine whether to accept or reject a specific quantity of material. Many authors studied about acceptance sampling plan based on life tests. Kantam and Rosaiah(1998) proposed half logistic distribution in acceptance sampling based on life tests, Kantam et al.(2001) developed acceptance sampling based on life tests: log-logistic model, Rosaiah et al.(2009) discussed Pareto distribution in acceptance sampling based on truncated life test, Srinivasa Rao et al.(2008) introduced acceptance sampling plans for Marshall-Olkin extended Lomax distribution. Baklizi(2003) proposed acceptance sampling based on truncated life tests in the Pareto distribution of the second kind. Srinivasa Rao and Kantam(2010) introduced acceptance sampling plans from truncated life tests based on log-logistic distribution for percentiles. Rosaiah et al.(2008) discussed an economic reliability test plan with Inverse Rayleigh variates. Krishna et.al (2013a,b) introduced Marshall-Olkin Frèchet distribution and its applications in reliability, sampling plan etc.

In this chapter a new distribution namely Negative Binomial Extreme Stable Marshall-Olkin Pareto distribution is introduced. The properties are also considered. A reliability test plan is developed for products with lifetime following the new distribution. Minimum sample size required is determined to assure a minimum average life needed when the life test is terminated at a pre assigned time t such that the observed number of failures does not exceed a given acceptance number c . The operating characteristic values and the minimum value of the ratio of true average life to required average life for various sampling plans are tabulated.

9.2 Negative Binomial Extreme Stable Marshall-Olkin Pareto Distribution

In this section we consider the survival function of Pareto distribution $\bar{F}(x, \sigma, \theta) = \left(\frac{x}{\sigma}\right)^{-\theta}; \sigma > 0$ and introduce a Negative Binomial Extreme Stable Marshall-Olkin Pareto distribution using negative binomial extreme stability. The survival function is given by

$$\bar{G}(x) = \frac{\alpha^\gamma}{1 - \alpha^\gamma} \left\{ \left[1 - \bar{\alpha} \left(\frac{x}{\sigma}\right)^{-\theta} \right]^{-\gamma} - 1 \right\}$$

for $x > 0, \alpha > 0, \sigma > 0, \theta > 0$ and $\bar{\alpha} = 1 - \alpha$.

Then the corresponding probability density function of the new distribution is given by

$$g(x; \alpha, \gamma, \sigma, \theta) = \frac{(1 - \alpha)\gamma\alpha^\gamma\theta\sigma^\theta x^{-(\theta+1)}}{(1 - \alpha^\gamma) \left[1 - \bar{\alpha} \left(\frac{x}{\sigma}\right)^{-\theta} \right]^{\gamma+1}}; \quad (9.2.1)$$

where $x > 0, \alpha > 0, \gamma > 0, \sigma > 0, \theta > 0$. We denote it as NBESMOP($\alpha, \gamma, \sigma, \theta$).

When $\gamma = 1$ then NBESMOP distribution reduces to Marshall-Olkin Pareto distribution. The cumulative distribution function is

$$G(x) = 1 - \frac{\alpha^\gamma}{1 - \alpha^\gamma} \left\{ \left[1 - \bar{\alpha} \left(\frac{x}{\sigma}\right)^{-\theta} \right]^{-\gamma} - 1 \right\} \quad (9.2.2)$$

Figure 9.1 shows the pdf of NBESMOP for different combinations of parameter values.

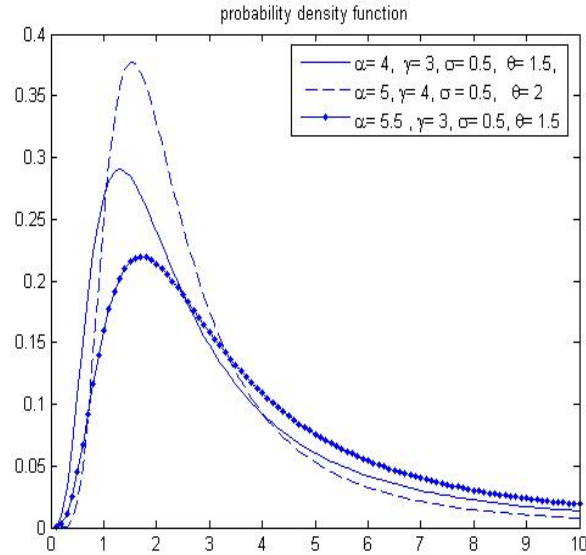


Figure 9.1: The p.d.f. of NBESMOP for different combinations of parameter value

The hazard rate function of the new distribution function is given by

$$h(x; \alpha, \gamma, \sigma, \theta) = \frac{(1 - \alpha)\gamma\alpha^\gamma\theta\sigma^\theta x^{-(\theta+1)}}{\left[1 - \bar{\alpha}\left(\frac{x}{\sigma}\right)^{-\theta}\right] \left\{1 - \left[1 - \bar{\alpha}\left(\frac{x}{\sigma}\right)^{-\theta}\right]^\gamma\right\}} \quad (9.2.3)$$

for $x > 0, \alpha > 0, \beta > 0, \gamma > 0$.

When $\gamma = 1$ the hazard rate function reduces to Marshall-Olkin Pareto distribution.

The p^{th} quantile function of the distribution, which is the inverse of the distribution function $F(x_p) = p$ is given by

$$x_p = \left[\frac{\bar{\alpha}\sigma^\theta [1 + \alpha^\gamma(1 - p)]^{\frac{1}{\gamma}}}{[1 + \alpha^\gamma(1 - p)]^{\frac{1}{\gamma}} + \alpha} \right]^{\frac{1}{\theta}}$$

Figure 9.2 shows the hazard rate function for different values of parameters.

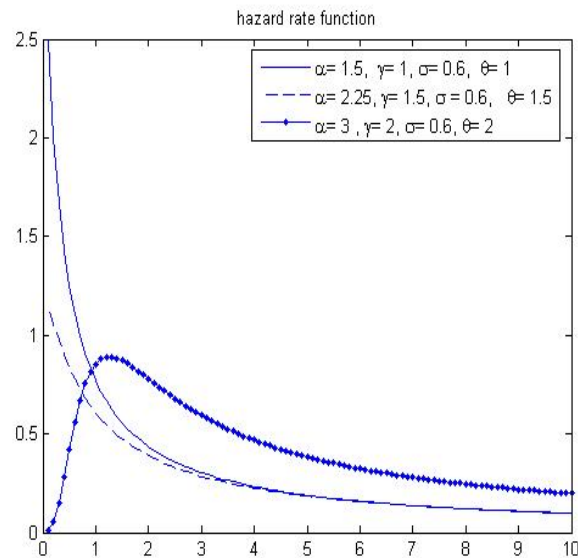


Figure 9.2: Hazard rate function of NBESMOP for different values of parameters

9.3 Reliability Test Plan

Assume that the life time of the product follows Negative Binomial Extreme Stable Marshall-Olkin Pareto distribution with scale parameter σ and the corresponding density function and cumulative distribution function is given in (9.2.1) and (9.2.2). In life testing experiments, the test is to terminate at a pre-determined time 't' and note the number of failures. The decision to accept the lot is made if and only if the number of observed failures at the end of the fixed time t does not exceed a given number c, which is called the acceptance number, with a given probability p^* . The test may be terminated before the time t is reached, when the already observed number of failures exceeds c, in which case the decision is to reject the lot. For such a truncated life test and the associated decision rule, we are interested in obtaining the smallest sample size necessary to achieve the objective.

In this distribution, the average life time depends only on σ , if α, γ and θ are known.

Let σ_0 be the required minimum average life time, then

$$G(x, \alpha, \gamma, \theta, \sigma) = G(x, \alpha, \gamma, \theta, \sigma_0) \iff \sigma \geq \sigma_0$$

A sampling plan consists of the following quantities:

1. The number of units n on test,
2. The acceptance number c ,
3. The maximum test duration t , and
4. The ratio t/σ_0 , where σ_0 is the specified average life.

The consumer's risk, i.e., the probability of accepting a bad lot should not exceed the value $1 - p^*$ where p^* is a lower bound for the probability that a lot of true value σ below σ_0 is rejected by the sampling plan. For fixed p^* the sampling plan is characterized by $(n, c, t/\sigma_0)$. For sufficiently large lots, we can apply binomial distribution for calculating the acceptance probability. The problem is to determine the smallest positive integer n for given values of c and t/σ_0 such that:

$$L(p_0) = \sum_{i=0}^c \binom{n}{i} p_0^i (1 - p_0)^{n-i} \leq 1 - p^* \quad (9.3.1)$$

where $p_0 = G(t; \alpha, \gamma, \theta, \sigma_0)$. The function $L(p)$ is the operating characteristic function of the sampling plan, i.e. the acceptance probability of the lot as a function of the failure probability $p(\sigma) = G(t; \alpha, \gamma, \theta, \sigma)$. The average lifetime of the products is increasing with σ and, therefore, the failure probability $p(\sigma)$ decreases with increasing σ implying that the operating characteristic function is increasing in σ . The minimum values of n satisfying (9.3.1) are obtained for $\alpha = 2, \gamma = 2, \theta = 2, p^* = 0.75, 0.90, 0.95, 0.99$ and $t/\sigma_0 = 2.214, 2.356, 2.556, 2.788, 3.102, 3.141, 3.408$. The results are given in Table 9.1.

If $p = G(t; \alpha, \gamma, \theta, \sigma)$ is small and n is large, the binomial probability may be approximated by Poisson probability with parameter $\lambda = np$ so that the left side of (9.3.1) can be written as

$$L^*(p_0) = \sum_{i=0}^c \frac{\lambda^i}{i!} e^{-\lambda} \leq 1 - p^* \quad (9.3.2)$$

where $\lambda = nG(t; \alpha, \gamma, \theta, \sigma_0)$. The minimum values of n satisfying (9.3.2) are obtained for the same combination of values as in the binomial case. The results are given in Table 9.2.

For a given value of p^* and t/σ_0 , the values of n and c are determined by means of the operating characteristic function. For some sampling plans, the values of the operating characteristic function depending on σ/σ_0 are displayed in Table 9.3.

The producers risk is the probability of rejection of the lot when $\sigma \geq \sigma_0$. We can compute the producers risk by first finding $p = G(t; \alpha, \gamma, \sigma, \theta)$ and then using the binomial distribution function. For a specified value of the producers risk say 0.05, one may be interested in knowing what value of σ or $\frac{\sigma}{\sigma_0}$ will ensure a producers risk less than or equal to 0.05 for a given sampling plan. The value of σ and, hence, the value of $\frac{\sigma}{\sigma_0}$ is the smallest positive number for which the following inequality holds

$$L(p_0) = \sum_{i=0}^c \binom{n}{i} p_0^i (1 - p_0)^{n-i} \geq 0.95 \quad (9.3.3)$$

For some sampling plans $(n, c, \frac{t}{\sigma_0})$ and values of p^* , minimum values of $\frac{\sigma}{\sigma_0}$ satisfying (9.3.3) are given in Table 9.4.

9.4 Description of the Tables and Their Application

Assume that the lifetime distribution is Negative Binomial Extreme Stable Marshall-Olkin Pareto distribution with $\alpha = 2, \theta = 2$ and $\gamma = 2$. Suppose that the experimenter is

interested in establishing that the true unknown average life is at least 1000 hours with confidence $p^* = 0.75$. It is desired to stop the experiment at $t = 2214$ hrs. Then, for an acceptance number $c = 2$, the required n in Table 9.1 is 6. If during 2214 hrs, no more than 2 failures out of 6 are observed, then the experimenter can assert, with a confidence level of 0.75 that the average life is at least 1000hrs. If the Poisson approximation to binomial probability is used, the value of n from Table 2 is 6. For this sampling plan ($n = 6, c = 2, t/\sigma_0 = 2.214$) under the Negative Binomial Extreme Stable Marshall-Olkin Pareto distribution the operating characteristic values from Table 9.3 are

$\frac{\sigma}{\sigma_0}$	1.2	1.4	1.6	1.8	2.0	2.2
L(p)	0.4149	0.653	0.8429	0.9527	0.9939	0.9999

From Table 9.4, we can get the value of $\frac{\sigma}{\sigma_0}$ for various choices of $c, \frac{\sigma}{\sigma_0}$ in order that the producers risk may not exceed 0.05. Thus in the above example the value of $\frac{\sigma}{\sigma_0}$ is 1.79 for $c = 2, \frac{\sigma}{\sigma_0} = 2.214, p^* = 0.75$. That is, the product should have an average life of 1.79 times the specified mean life of 1000 hrs in order that under the above acceptance sampling plan, the product is accepted with probability 0.95. The actual average life necessary to accept 95% of the lots is provided by Table 9.4.

Consider the following ordered failure times of the release of a software in terms of hours from starting of the execution of the software up to the time at which a failure of the software occurs (Wood(1996)). This data can be regarded as an ordered sample of size $n = 16$ with observations $\{x_i, i = 1, 2, \dots, 16\} = \{519, 968, 1430, 1893, 2490, 3058, 3625, 4422, 5218, 5823, 6539, 7083, 7487, 7846, 8205, 8564\}$.

Let the required average life time be 1000 hours and the testing time be $t = 2214$ hours. This leads to the ratio $\frac{t}{\sigma_0} = 2.214$ with a corresponding sample size $n = 16$ and an acceptance number $c=4$, which can be obtained from Table 9.1 for $p^* = 0.99$. The sampling plan for the above sample data is ($n = 16, c = 4, \frac{t}{\sigma_0} = 2.214$). Based on the observations we have to decide whether to accept the product or reject it. We accept the product only if the number of failures before 2214 hours is less than or equal to 4. In the

sample of 16 failure time instants, there is 4 failure hours before the termination $t = 2214$ hours. Hence we accept the product.

9.5 Conclusion

In this chapter, we have considered the generalization of the Marshall-Olkin family of distributions using negative binomial compounding instead of geometric compounding. The newly obtained generalized distribution is the Negative Binomial Extreme Stable Marshall-Olkin Pareto distribution. Also a reliability test plan is developed when the life times of the items follow NBESMOP distribution. The results are illustrated by a numerical example. Table 9.1 gives minimum sample size for the specified ratio $\frac{t}{\sigma_0}$, confidence level p^* , acceptance number c , $\alpha = 2, \theta = 2$ and $\gamma = 2$ using binomial approximation. Table 9.2 gives minimum sample size for the specified ratio $\frac{t}{\sigma_0}$, confidence level p^* , acceptance number c , $\alpha = 2, \theta = 2$ and $\gamma = 2$ using Poisson approximation. Table 9.3 gives values of the operating characteristic function of the sampling plan $(n, c, \frac{t}{\sigma_0})$ for given confidence level p^* with $\alpha = 2, \theta = 2$ and $\gamma = 2$. Table 9.4 gives Minimum ratio of true σ and required σ_0 for the acceptability of a lot with producer's risk of 0.05 for $\alpha = 2, \theta = 2$ and $\gamma = 2$.

Table 9.1: Minimum sample size for the specified ratio t/σ_0 , confidence level p^* , acceptance number c , $\alpha = 2, \theta = 2$ and $\gamma = 2$ using binomial approximation.

p^*	c	t/σ_0						
		2.214	2.356	2.556	2.788	3.102	3.141	3.408
.75	0	3	2	2	2	1	1	1
.75	1	4	4	4	3	3	3	3
.75	2	6	6	5	5	4	4	4
.75	3	8	7	7	6	6	6	6
.75	4	10	9	9	8	7	7	7
.75	5	12	11	10	9	9	9	8
.75	6	14	13	12	11	10	10	10
.75	7	15	14	13	12	12	11	11
.75	8	17	16	15	14	13	13	12
.75	9	19	18	17	15	14	14	14
.75	10	21	20	18	17	16	16	15
.90	0	4	3	3	2	2	2	2
.90	1	5	5	5	4	4	4	4
.90	2	8	7	6	6	5	5	5
.90	3	10	9	8	8	7	7	7
.90	4	12	11	10	9	8	8	8
.90	5	14	13	12	11	10	10	9
.90	6	16	15	13	12	11	11	11
.90	7	18	17	15	14	13	13	12
.90	8	20	18	17	16	14	14	14
.90	9	22	20	19	17	16	16	15
.90	10	24	22	20	19	17	17	16
.95	0	4	4	3	3	3	3	2
.95	1	6	6	5	5	4	4	4
.95	2	9	8	7	7	6	6	6
.95	3	11	10	9	8	8	8	7
.95	4	13	12	11	10	9	9	9
.95	5	15	14	13	12	11	11	10
.95	6	17	16	15	13	12	12	12
.95	7	19	18	16	15	14	14	13
.95	8	21	20	18	17	15	15	14
.95	9	23	22	20	18	17	17	16
.95	10	25	24	22	20	18	18	17
.99	0	6	5	5	4	4	4	3
.99	1	9	8	7	6	6	6	5
.99	2	11	10	9	8	8	7	7
.99	3	14	12	12	10	9	9	9
.99	4	16	15	13	12	11	11	10
.99	5	18	17	15	14	13	12	12
.99	6	20	19	17	16	14	14	13
.99	7	23	21	19	17	16	16	15
.99	8	25	23	21	19	17	17	16
.99	9	27	25	23	21	19	19	18
.99	10	29	27	24	22	21	20	19

Table 9.2: Minimum sample size for the specified ratio t/σ_0 , confidence level p^* , acceptance number c , $\alpha = 2, \theta = 2$ and $\gamma = 2$ using Poisson approximation.

p^*	c	t/σ_0						
		2.214	2.356	2.556	2.788	3.102	3.141	3.408
0.75	0	2	2	2	2	1	1	1
0.75	1	4	4	4	3	3	3	3
0.75	2	6	6	5	5	4	4	4
0.75	3	8	7	7	6	6	6	6
0.75	4	11	9	9	8	7	7	7
0.75	5	12	11	10	9	9	9	8
0.75	6	14	13	12	11	10	10	10
0.75	7	15	14	13	12	12	11	11
0.75	8	17	16	15	14	13	13	12
0.75	9	19	18	17	15	14	14	14
0.75	10	21	20	18	17	16	16	15
0.9	0	3	3	3	2	2	2	2
0.9	1	5	5	5	4	4	4	4
0.9	2	8	7	6	6	5	5	5
0.9	3	10	9	8	8	7	7	7
0.9	4	12	11	10	9	8	8	8
0.9	5	14	13	12	11	10	10	9
0.9	6	16	15	13	12	11	11	11
0.9	7	18	17	15	14	13	13	12
0.9	8	20	18	17	16	14	14	14
0.9	9	22	20	19	17	16	16	15
0.9	10	24	22	20	19	17	17	16
0.95	0	4	4	3	3	3	3	2
0.95	1	6	6	5	5	4	4	4
0.95	2	9	8	7	7	6	6	6
0.95	3	11	10	9	8	8	8	7
0.95	4	13	12	11	10	9	9	9
0.95	5	15	14	13	12	11	11	10
0.95	6	17	16	15	13	12	12	12
0.95	7	19	18	16	15	14	14	13
0.95	8	21	20	18	17	15	15	14
0.95	9	23	22	20	18	17	17	16
0.95	10	25	24	22	20	18	18	17
0.99	0	6	5	5	4	4	4	3
0.99	1	9	8	7	6	6	6	5
0.99	2	11	10	9	8	8	7	7
0.99	3	14	12	11	10	9	9	9
0.99	4	16	15	13	12	11	11	10
0.99	5	18	17	15	14	13	12	12
0.99	6	20	19	17	16	14	14	13
0.99	7	23	21	19	17	16	16	15
0.99	8	25	23	21	19	17	17	16
0.99	9	27	25	23	21	19	19	18
0.99	10	29	27	24	22	21	20	19

Table 9.3: Values of the operating characteristic function of the sampling plan $(n, c, t/\sigma_0)$ for a given p^* with $\alpha = 2, \theta = 2$ and $\gamma = 2$.

p^*	n	c	t/σ_0	σ/σ_0					
				1.2	1.4	1.6	1.8	2.0	2.2
0.75	6	2	2.214	0.4149	0.653	0.8429	0.9527	0.9939	0.9999
0.75	6	2	2.356	0.3302	0.5549	0.7605	0.9032	0.9756	0.9981
0.75	5	2	2.556	0.3852	0.5828	0.7575	0.8829	0.9565	0.9897
0.75	5	2	2.788	0.2901	0.468	0.6453	0.7936	0.8987	0.9607
0.75	4	2	3.102	0.3916	0.5472	0.6897	0.8056	0.891	0.9452
0.75	4	2	3.141	0.3801	0.534	0.6766	0.7942	0.8813	0.9393
0.75	4	2	3.408	0.3097	0.4499	0.5895	0.7143	0.8157	0.8909
0.9	8	2	2.214	0.1982	0.4348	0.6982	0.8949	0.9847	.9999
0.9	7	2	2.356	0.2148	0.431	0.6671	0.8549	0.961	0.9968
0.9	6	2	2.556	0.2359	0.4302	0.6371	0.8103	0.9246	0.9811
0.9	6	2	2.788	0.1581	0.3123	0.5004	0.6852	0.8339	0.9314
0.9	5	2	3.102	0.197	0.3402	0.5028	0.6601	0.7922	0.8891
0.9	5	2	3.141	0.1877	0.3267	0.4865	0.6435	0.7776	0.878
0.9	5	2	3.408	0.1351	0.2465	0.3852	0.5341	0.6749	0.7936
0.95	9	2	2.214	0.1316	0.3441	0.6229	0.8592	0.9783	0.9999
0.95	8	2	2.356	0.1351	0.3261	0.5743	0.8007	0.9429	0.9951
0.95	7	2	2.556	0.1379	0.3056	0.5201	0.73	0.8852	0.9695
0.95	7	2	2.788	0.0818	0.1995	0.3747	0.5763	0.7607	0.895
0.95	6	2	3.102	0.0917	0.1977	0.3465	0.5176	0.6831	0.8194
0.95	6	2	3.141	0.0857	0.1867	0.3303	0.4983	0.6638	0.8032
0.95	6	2	3.408	0.0543	0.1254	0.2359	0.3785	0.5352	0.685
0.99	11	2	2.214	0.0548	0.2058	0.4801	0.7789	0.9617	0.9999
0.99	10	2	2.356	0.0498	0.1756	0.4067	0.683	0.8975	0.9902
0.99	9	2	2.556	0.0428	0.1417	0.3259	0.5657	0.7905	0.9377
0.99	8	2	2.788	0.0407	0.1234	0.2728	0.4743	0.6838	0.8529
0.99	8	2	3.102	0.0171	0.0584	0.1465	0.2891	0.4717	0.6613
0.99	7	2	3.141	0.037	0.1015	0.2149	0.3724	0.5514	0.7208
0.99	7	2	3.408	0.0206	0.0605	0.1379	0.2575	0.4104	0.57608

Table 9.4: Minimum ratio of true σ and required σ_0 for the acceptability of a lot with producer's risk of 0.05 for $\alpha = 2, \theta = 2$ and $\gamma = 2$

p*	c	t/σ_0						
		2.214	2.356	2.556	2.788	3.102	3.141	3.408
.75	0	2.16	2.27	2.46	2.69	2.88	2.92	3.17
.75	1	1.93	2.05	2.23	2.31	2.58	2.61	2.83
.75	2	1.79	1.91	1.98	2.16	2.24	2.26	2.46
.75	3	1.71	1.75	1.90	1.94	2.16	2.19	2.38
.75	4	1.65	1.69	1.83	1.90	1.98	2.00	2.17
.75	5	1.60	1.64	1.72	1.78	2.00	2.03	2.04
.75	6	1.56	1.62	1.69	1.76	1.86	1.89	2.04
.75	7	1.51	1.54	1.61	1.67	1.88	1.78	1.93
.75	8	1.47	1.52	1.59	1.67	1.78	1.81	1.96
.75	9	1.45	1.50	1.59	1.60	1.78	1.81	1.87
.75	10	1.45	1.50	1.53	1.62	1.71	1.73	1.80
.90	0	2.17	2.30	2.50	2.69	2.99	3.03	3.28
.90	1	1.99	2.12	2.30	2.45	2.73	2.76	3.00
.90	2	1.89	1.90	2.08	2.27	2.43	2.46	2.67
.90	3	1.80	1.89	1.97	2.15	2.31	2.33	2.53
.90	4	1.73	1.82	1.90	2.02	2.14	2.16	2.35
.90	5	1.69	1.75	1.85	1.90	2.08	2.11	2.17
.90	6	1.65	1.71	1.76	1.85	1.98	2.00	2.17
.90	7	1.60	1.66	1.74	1.83	1.95	1.98	2.07
.90	8	1.58	1.60	1.69	1.80	1.86	1.87	2.04
.90	9	1.54	1.58	1.67	1.73	1.86	1.87	1.96
.90	10	1.52	1.56	1.63	1.73	1.81	1.83	1.88
.95	0	2.18	2.32	2.50	2.73	3.04	3.08	3.30
.95	1	2.03	2.16	2.31	2.52	2.73	2.76	3.00
.95	2	1.93	2.02	2.16	2.34	2.52	2.55	2.77
.95	3	1.85	1.92	2.05	2.15	2.40	2.43	2.53
.95	4	1.78	1.84	1.97	2.07	2.22	2.25	2.44
.95	5	1.71	1.8	1.90	2.02	2.19	2.22	2.29
.95	6	1.67	1.75	1.85	1.92	2.06	2.08	2.24
.95	7	1.63	1.71	1.78	1.90	2.03	2.06	2.15
.95	8	1.60	1.69	1.74	1.85	1.95	1.98	2.07
.95	9	1.58	1.64	1.72	1.80	1.93	1.95	2.04
.95	10	1.54	1.62	1.69	1.78	1.86	1.88	1.99
.99	0	2.19	2.32	2.52	2.74	3.05	3.09	3.33
.99	1	2.09	2.21	2.34	2.56	2.84	2.88	3.08
.99	2	1.99	2.10	2.25	2.39	2.66	2.65	2.88
.99	3	1.92	1.99	2.16	2.27	2.45	2.52	2.73
.99	4	1.85	1.94	2.05	2.18	2.40	2.39	2.53
.99	5	1.80	1.89	1.97	2.13	2.31	2.28	2.47
.99	6	1.75	1.84	1.92	2.07	2.19	2.22	2.35
.99	7	1.73	1.80	1.88	1.99	2.16	2.19	2.32
.99	8	1.69	1.75	1.85	1.94	2.08	2.11	2.20
.99	9	1.67	1.73	1.83	1.92	2.03	2.06	2.17
.99	10	1.63	1.71	1.76	1.85	2.03	2.00	2.14

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CHAPTER 10

Negative Binomial Marshall- Olkin Rayleigh Distribution and its Applications

10.1 Introduction

Rayleigh distribution is one of the most popular distributions in analyzing skewed data. The Rayleigh distribution was originally proposed in the fields of acoustics and optics. It is used in oceanography and in communication theory for describing instantaneous peak power of received radio signals. This distribution has also been applied in several areas such as health, agriculture, biology etc. Rayleigh distribution is a special case of the two-parameter Weibull distribution with the shape parameter equal to two. Surles and Padgett(2001) introduced two-parameter Burr Type X distribution and named it as the generalized Rayleigh distribution. The two-parameter generalized Rayleigh distribution is a particular member of the generalized Weibull distribution, proposed by Mudholkar and Srivastava(1993). Cordeiro et al. (2013) and Gomes et al. (2014) introduced two general-

izations for the Generalized Rayleigh distribution: the four-parameter beta-GR distribution and the four-parameter Kumaraswamy-GR distribution. Many authors have proposed various univariate distributions belonging to the Marshall-Olkin family of distributions such as Alice and Jose(2003), Ghitany et al.(2007), Jayakumar and Thomas(2008), Garcia et al.(2010), Ghitany et al.(2012), Krishna et al (2013). General properties of the MOE family of distributions were studied recently by Barreto-Souza et al. (2013) and Cordeiro et al. (2014).

Acceptance sampling plan is an inspection procedure used to determine whether to accept or reject a specific quantity of material. Many authors studied about acceptance sampling plan based on life tests. Kantam and Rosaiah(1998) proposed half logistic distribution in acceptance sampling based on life tests, Kantam et al.(2001) developed acceptance sampling based on life tests: log-logistic model, Rosaiah et al.(2009) discussed Pareto distribution in acceptance sampling based on truncated life test, Srinivasa Rao et al.(2008) introduced acceptance sampling plans for Marshall-Olkin extended Lomax distribution. Rosaiah et al.(2005) discussed an acceptance sampling plan based on the Inverse Rayleigh distribution. Krishna et.al (2013a,b) introduced Marshall-Olkin Frèchet distribution and its applications in reliability, sampling plan etc. Tsai and Wu (2006) considered the problem of acceptance sampling plan based on truncated life tests when the lifetime of product follows the generalized Rayleigh distribution for known shape parameter. Kantam et al.(2013) proposed economic reliability test plans based on Rayleigh distribution. Singh et al.(2013) discussed acceptance sampling plan based on truncated life tests for compound Rayleigh distribution.

In this chapter we discuss Negative Binomial Marshall-Olkin Rayleigh Distribution and its Applications. Quantiles and order statistics of the distribution are obtained. We developed the reliability test plan for the distribution. Minimum sample size required is determined to assure a minimum average life needed when the life test is terminated at a pre-assigned time t such that the observed number of failures does not exceed a given acceptance number c . The operating characteristic values and the minimum value of the

ratio of true average life to required average life for various sampling plans are tabulated.

10.2 Negative Binomial Marshall-Olkin Rayleigh Distribution

In this section we consider the survival function of Rayleigh distribution $\bar{F}(x, \sigma) = \exp\left(\frac{-x^2}{2\sigma^2}\right)$. From negative binomial extreme stable family, we introduce a Negative Binomial Marshall-Olkin Rayleigh distribution given by the survival function

$$\bar{G}(x) = \frac{\alpha^\gamma}{1 - \alpha^\gamma} \left\{ \left[1 - \bar{\alpha} \exp\left(\frac{-x^2}{2\sigma^2}\right) \right]^{-\gamma} - 1 \right\} \quad (10.2.1)$$

for $x > 0, \alpha > 0, \sigma > 0, \gamma > 0$ and $\bar{\alpha} = 1 - \alpha$.

Then the corresponding probability density function of the new distribution is

$$g(x; \alpha, \gamma, \sigma) = \frac{(1 - \alpha)\gamma\alpha^\gamma \frac{x}{\sigma^2} \exp\left(\frac{-x^2}{2\sigma^2}\right)}{(1 - \alpha^\gamma) \left[1 - \bar{\alpha} \exp\left(\frac{-x^2}{2\sigma^2}\right) \right]^{\gamma+1}}; \quad (10.2.2)$$

where $x > 0, \alpha > 0, \gamma > 0, \sigma > 0$. We denote it as NBMOR(α, γ, σ).

When $\gamma = 1$ then NBMOR distribution reduces to Marshall-Olkin Rayleigh distribution. The cumulative distribution function is

$$G(x) = 1 - \frac{\alpha^\gamma}{1 - \alpha^\gamma} \left\{ \left[1 - \bar{\alpha} \exp\left(\frac{-x^2}{2\sigma^2}\right) \right]^{-\gamma} - 1 \right\} \quad (10.2.3)$$

Figure 10.1 shows the pdf of NBMOR for different combinations of parameter values.

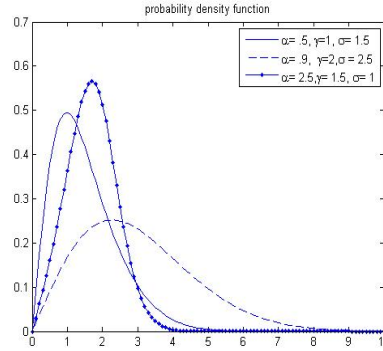


Figure 10.1: The p.d.f. of NBMOR for different values of parameters

The hazard rate function of the new distribution function is

$$h(x; \alpha, \gamma, \sigma) = \frac{(1 - \alpha)\gamma \frac{x}{\sigma^2} \exp\left(\frac{-x^2}{2\sigma^2}\right)}{\left[1 - \bar{\alpha} \exp\left(\frac{-x^2}{2\sigma^2}\right)\right] \left[1 - (1 - \bar{\alpha} \exp\left(\frac{-x^2}{2\sigma^2}\right))^\gamma\right]} \quad (10.2.4)$$

for $x > 0, \alpha > 0, \gamma > 0, \sigma > 0$.

When $\gamma = 1$ the hazard rate function reduces to Marshall-Olkin Rayleigh distribution.

10.3 Quantiles and Order Statistics

The p^{th} quantile function of the distribution, the inverse of the distribution function $F(x_p) = p$, is given by

$$x = \left(2\sigma^2 \log \left[\frac{\bar{\alpha}(1 - p + p\alpha^\gamma)^{\frac{1}{\gamma}}}{(1 - p + p\alpha^\gamma)^{\frac{1}{\gamma}} - \alpha} \right] \right)^{\frac{1}{2}}$$

Let X_1, X_2, \dots, X_n be a random sample taken from the NBMOR distribution and $X_{1:n}, X_{2:n}, \dots, X_{n:n}$ be the corresponding order statistics. The survival function of NBMOR

distribution is given by (10.2.1). Then the c.d.f. of the first order statistic $X_{1:n}$ is given by

$$\begin{aligned} G_{1:n}(x) &= 1 - (\bar{G}(x))^n \\ &= 1 - \frac{\alpha^{n\gamma}(-1)^n}{(1 - \alpha^\gamma)^n} \sum_{k=0}^n (-1)^k \binom{n}{k} \left[1 - \bar{\alpha} \exp\left(\frac{-x^2}{2\sigma^2}\right) \right]^{-\gamma k} \end{aligned}$$

The c.d.f. of the n^{th} order statistic $X_{n:n}$ is given by

$$\begin{aligned} G_{n:n}(x) &= [1 - \bar{G}(x)]^n \\ &= \frac{1}{(1 - \alpha^\gamma)^n} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\alpha^{\gamma k}}{\left[1 - \bar{\alpha} \exp\left(\frac{-x^2}{2\sigma^2}\right) \right]^{-\gamma k}} \end{aligned}$$

The probability density function $g_{i:n}(x)$ of the i^{th} order statistics $X_{i:n}$ is given by

$$\begin{aligned} g_{i:n}(x; \alpha, \gamma, \sigma) &= \frac{n!}{(i-1)!(n-1)!} \frac{(1-\alpha)\gamma\alpha^\gamma \frac{x}{\sigma^2} \exp\left(\frac{-x^2}{2\sigma^2}\right)}{(1-\alpha^\gamma) \left[1 - \bar{\alpha} \exp\left(\frac{-x^2}{2\sigma^2}\right) \right]^{\gamma+1}} \\ &\quad \left[1 - \frac{\alpha^\gamma}{1-\alpha^\gamma} \left\{ \left[1 - \bar{\alpha} \exp\left(\frac{-x^2}{2\sigma^2}\right) \right]^{-\gamma} - 1 \right\} \right]^{i-1} \\ &\quad \left[\frac{\alpha^\gamma}{1-\alpha^\gamma} \left\{ \left[1 - \bar{\alpha} \exp\left(\frac{-x^2}{2\sigma^2}\right) \right]^{-\gamma} - 1 \right\} \right]^{n-i} \end{aligned}$$

This can be written as a finite mixture of the probability density functions of NBMOR distributed random variables since

$$\begin{aligned} g_{i:n}(x; \alpha, \gamma, \sigma) &= \frac{n!(-1)^{n-i}}{(i-1)!(n-1)!} \frac{\alpha^{\gamma(n-i)}}{(1-\alpha^\gamma)^n} \sum_{r=0}^{i-1} \sum_{s=0}^{n-i} \binom{i-1}{r} \binom{n-i}{s} \\ &\quad \frac{(-1)^{r+s} [1 - \alpha^{\gamma(r+s+1)}]}{\alpha^s (r+s+1)} g(x; \alpha, \gamma(r+s+1), \sigma) \end{aligned}$$

10.4 Estimation of Parameters

In this section we consider maximum likelihood estimation with respect to a given sample (x_1, x_2, \dots, x_n) . Then the log likelihood function is given by

$$\begin{aligned} \log L = & n \log(1 - \alpha) + n \log \gamma + n \gamma \log \alpha + \sum_{i=0}^n \log\left(\frac{x_i}{\sigma^{2n}}\right) + \sum_{i=0}^n \frac{x_i^2}{2\sigma^2} - \\ & n \log(1 - \alpha^\gamma) - (\gamma + 1) \sum_{i=0}^n \log\left[1 - \bar{\alpha} \exp\left(\frac{-x_i^2}{2\sigma^2}\right)\right] \end{aligned}$$

Taking partial derivatives with respect to parameters we get,

$$\frac{\partial \log L}{\partial \alpha} = \frac{-n}{1 - \alpha} + \frac{n\gamma}{\alpha} - \frac{n\gamma\alpha^{\gamma-1}}{1 - \alpha^\gamma} - (\gamma + 1) \sum_{i=0}^n \frac{\exp\left(\frac{-x_i^2}{2\sigma^2}\right)}{[1 - \bar{\alpha} \exp\left(\frac{-x_i^2}{2\sigma^2}\right)]}$$

$$\frac{\partial \log L}{\partial \gamma} = \frac{n}{\gamma} + n \log(\alpha) - \frac{n}{1 - \alpha^\gamma} \alpha^\gamma \log(\alpha) + \sum_{i=0}^n \log\left[1 - \bar{\alpha} \exp\left(\frac{-x_i^2}{2\sigma^2}\right)\right]$$

$$\frac{\partial \log L}{\partial \sigma} = \frac{-2n^2}{\sigma} + \sum_{i=0}^n \frac{x_i^2}{\sigma^3} - (\gamma + 1) \sum_{i=0}^n \frac{\bar{\alpha} x_i^2 \exp\left(\frac{-x_i^2}{2\sigma^2}\right)}{\sigma^3 [1 - \bar{\alpha} \exp\left(\frac{-x_i^2}{2\sigma^2}\right)]}$$

The maximum likelihood estimates can be obtained by solving the equations

$\frac{\partial \log L}{\partial \alpha} = 0, \frac{\partial \log L}{\partial \gamma} = 0, \frac{\partial \log L}{\partial \sigma} = 0$. The equations can be solved using nlm function in R software.

Table 10.1: Estimates, AIC , BIC and K-S statistic for the data set

Model	Parameter	Estimates	AIC	BIC	K-S statistic
NBMOR	α	0.1019	2571.586	2526.142	0.7532
	γ	0.0992			
	σ	0.0619			
Rayleigh	σ	0.0019	2617.55	2614.402	0.8914

10.4.1 Data Analysis

In this section we consider a data analysis. The data represents an uncensored data set corresponding to remission times (in months) of a random sample of 128 bladder cancer patients reported in Lee and Wang(2003). 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69, 5.49

We compare the goodness of fit of NBMOR and Rayleigh distributions. For that we estimate the parameters by the the method of maximum likelihood and fit both distributions to the data. The maximum likelihood estimates, AIC and BIC values, and K-S statistic are given in Table 10.1. From Table 10.1, it is seen that the smallest AIC and BIC values are obtained in NBMOR distribution. Hence we conclude that NBMOR distribution is a better model for the data set.

10.5 Reliability Test Plan

Let a lot of products of infinitely large size be submitted for sampling inspection and decision to reject or accept. Assume that the life time of the product follows Negative Binomial Marshall-Olkin Rayleigh distribution with scale parameter σ and the corresponding density function and cumulative distribution function is given in (10.2.2) and (10.2.3). In life testing experiment the test is to terminate at a predetermined time 't' and note the number of failures. The decision to accept the lot is made if and only if the number of observed failures at the end of the fixed time t does not exceed a given number c, which is called the acceptance number, with a given probability p^* . The test may be terminated before the time t is reached, when the already observed number of failures exceeds c, in which case the decision is to reject the lot. For such a truncated life test and the associated decision rule, we are interested in obtaining the smallest sample size necessary to achieve the objective.

In this distribution the average life time depends only on σ , if α and γ are known. Let σ_0 be the required minimum value of σ , then

$$G(x, \alpha, \gamma, \sigma) = G(x, \alpha, \gamma, \sigma_0) \iff \sigma \geq \sigma_0$$

A sampling plan consists of the following quantities:

1. The number of units n on test,
2. The acceptance number c,
3. The maximum test duration t, and
4. The ratio t/σ_0 , where σ_0 is the specified average life.

The consumer's risk, i.e., the probability of accepting a bad lot should not exceed the value $1 - p^*$ with p^* is a lower bound for the probability that a lot of true value σ below

σ_0 is rejected by the sampling plan. For fixed p^* the sampling plan is characterized by $(n, c, t/\sigma_0)$. By sufficiently large lots we can apply binomial distribution for calculating the acceptance probability. The problem is to determine the smallest positive integer n for given values of c and t/σ_0 such that:

$$L(p_0) = \sum_{i=0}^c \binom{n}{i} p_0^i (1 - p_0)^{n-i} \leq 1 - p^* \quad (10.5.1)$$

where $p_0 = G(t; \alpha, \gamma, \sigma_0)$. The function $L(p)$ is the operating characteristic function of the sampling plan, i.e. the acceptance probability of the lot as a function of the failure probability $p(\sigma) = G(t; \alpha, \gamma, \sigma)$. The average lifetime of the products is increasing with σ and, therefore, the failure probability $p(\sigma)$ decreases with increasing σ implying that the operating characteristic function is increasing in σ . The minimum values of n satisfying (10.5.1) are obtained for $\alpha = 2, \gamma = 2, p^* = 0.75, 0.90, 0.95, 0.99$ and $t/\sigma_0 = 1.0, 1.25, 1.50, 1.75, 2.0, 2.5, 3.0$. The results are given in Table 10.2.

If $p = G(t; \alpha, \gamma, \sigma)$ is small and n is very large, the binomial probability may be approximated by Poisson probability with parameter $\lambda = np$ so that the left side of (10.5.1) can be written as

$$L^*(p_0) = \sum_{i=0}^c \frac{\lambda^i}{i!} e^{-\lambda} \leq 1 - p^* \quad (10.5.2)$$

where $\lambda = nG(t; \alpha, \gamma, \sigma_0)$. The minimum values of n satisfying (10.5.2) are obtained for the same combination of values as in the binomial case. The results are given in Table 10.3.

For a given value of p^* and t/σ_0 , the values of n and c are determined by means of the operating characteristic function. For some sampling plans, the values of the operating characteristic function depending on σ/σ_0 are displayed in Table 10.4.

The producers risk is the probability of rejection of the lot when $\sigma \geq \sigma_0$. We can compute the producers risk by first finding $p = G(t; \alpha, \gamma, \sigma)$ and then using the binomial distribution function. For a specified value of the producers risk say 0.05, one may be interested in knowing what value of σ or $\frac{\sigma}{\sigma_0}$ will ensure a producers risk less than or equal to 0.05 for a given sampling plan. The value σ and, hence, the value of $\frac{\sigma}{\sigma_0}$ is the smallest positive number for which the following inequality holds

$$L(p_0) = \sum_{i=0}^c \binom{n}{i} p_0^i (1 - p_0)^{n-i} \geq 0.95 \tag{10.5.3}$$

For some sampling plan $(n, c, \frac{t}{\sigma_0})$ and values of p^* , minimum values of $\frac{\sigma}{\sigma_0}$ satisfying (10.5.3) are given in Table 10.5.

10.6 Description of the Tables and their Application

Assume that the lifetime distribution is Negative Binomial Marshall-Olkin Rayleigh distribution with $\alpha = 2$ and $\gamma = 2$. Suppose that the experimenter is interested in establishing that the true unknown average life is at least 1000 hours with confidence $p^* = 0.75$. It is desired to stop the experiment at $t = 1000$ hrs. Then, for an acceptance number $c = 2$, the required n in Table 10.2 is 21. If during 1000hrs, no more than 2 failures out of 21 are observed, then the experimenter can assert, with a confidence level of 0.75 that the average life is at least 1000hrs. If the Poisson approximation to binomial probability is used, the value of n from Table 10.3 is 22. For this sampling plan $(n = 21, c = 2, t/\sigma_0 = 1.00)$ under the Negative Binomial Marshall-Olkin Rayleigh distribution the operating characteristic values from Table 10.4 are

$\frac{\sigma}{\sigma_0}$	2	4	6	8	10	12
L(p)	0.7061	0.9872	0.9981	0.9995	0.9998	0.9999

From Table 10.5, we can get the value of $\frac{\sigma}{\sigma_0}$ for various choices of $c, \frac{\sigma}{\sigma_0}$ in order that the producers risk may not exceed 0.05. Thus in the above example the value of $\frac{\sigma}{\sigma_0}$ is 0.89 for

$c = 2$, $\frac{\sigma}{\sigma_0} = 1.0$, $p^* = 0.75$. That is, the product should have an average life of 0.89 times the specified mean life of 1000hrs in order that under the above acceptance sampling plan, the product is accepted with probability 0.95. The actual average life necessary to accept 95% of the lots is provided by Table 10.5.

Consider the following ordered failure times of the release of a software given in terms of hours from starting of the execution of the software upto the time at which a failure of the software occurs (Wood, 1996). This data can be regarded as an ordered sample of size $n = 11$ with observations with observations $\{x_i, i = 1, 2, \dots, 11\} = \{519, 968, 1430, 1893, 2490, 3058, 3625, 4422, 5218, 5823, 6539\}$

Let the required average lifetime be 1000 hours and the testing time be $t = 1500$ hours, which leads to a sample size $n = 11$ with a corresponding ratio of $\frac{t}{\sigma_0} = 1.5$ and an acceptance number $c = 3$ which are obtained from Table 10.2 for $p^* = 0.75$. Therefore, the sampling plan for the above sample data is $(n = 11, c = 3, \frac{t}{\sigma_0} = 1.5)$. Based on the observations, we have to decide whether to accept the product or reject it. We accept the product only, if the number of failures before 1500 hours is less than or equal to 3. From the given ordered sample we notice that the earliest failures of the software product are at 519 , 968 and 1430 hours, which are less than 1500 hours. Therefore we accept the product.

10.7 Conclusion

In this chapter we consider the Negative Binomial Marshall-Olkin Rayleigh distribution. Properties of the new distribution are investigated. Maximum likelihood estimates are obtained. The use of the model in lifetime modeling is established by fitting it to a real data set on remission times of bladder cancer patients. The results are given in Table 10.1. A reliability test plan is developed when the life times of the items follow NBMOR distribution. The results are illustrated by a numerical example. Table 10.2 gives minimum sample size for the specified ratio $\frac{t}{\sigma_0}$, confidence level p^* , acceptance number c , $\alpha = 2, \gamma = 2$ using binomial approximation. Table 10.3 gives minimum sample size for the specified ratio $\frac{t}{\sigma_0}$,

Table 10.2: Minimum sample size for the specified ratio t/σ_0 , confidence level p^* , acceptance number c , $\alpha = 2, \gamma = 2$ using binomial approximation.

p^*	c	t/σ_0						
		1.00	1.25	1.50	1.75	2.00	2.50	3.00
.75	0	7	4	3	2	2	1	1
.75	1	14	9	6	4	3	2	2
.75	2	21	13	9	6	5	4	3
.75	3	27	17	11	8	7	5	4
.75	4	34	21	14	10	8	6	5
.75	5	40	25	17	12	10	7	6
.75	6	46	28	19	14	11	8	7
.75	7	52	32	22	16	13	10	8
.75	8	58	36	24	18	14	11	9
.75	9	64	40	27	20	16	12	11
.75	10	70	43	29	22	17	13	12
.90	0	12	7	5	3	2	2	1
.90	1	20	12	8	6	4	3	2
.90	2	28	17	11	8	6	4	3
.90	3	35	21	14	10	8	5	5
.90	4	24	26	17	12	9	7	6
.90	5	49	30	20	14	11	8	7
.90	6	56	34	23	16	13	9	8
.90	7	62	38	25	18	14	10	9
.90	8	69	42	28	20	16	12	10
.90	9	75	46	31	22	18	13	11
.90	10	82	50	34	24	19	14	12
.95	0	15	9	6	4	3	2	1
.95	1	24	15	10	7	5	3	3
.95	2	33	20	13	9	7	5	4
.95	3	40	24	16	11	9	6	5
.95	4	48	29	19	14	10	7	6
.95	5	55	33	22	16	12	8	7
.95	6	62	38	25	18	14	10	8
.95	7	69	42	28	20	15	11	9
.95	8	76	46	31	22	17	12	10
.95	9	83	50	34	24	19	13	11
.95	10	89	54	36	26	20	15	12
.99	0	23	14	9	6	4	3	2
.99	1	34	20	13	9	7	4	3
.99	2	43	26	17	12	9	6	4
.99	3	52	31	20	14	11	7	5
.99	4	60	36	24	17	12	8	7
.99	5	68	41	27	19	14	10	8
.99	6	76	45	30	21	16	11	9
.99	7	83	50	33	23	18	12	10
.99	8	91	55	36	26	20	13	11
.99	9	99	59	39	28	21	15	12
.99	10	105	64	42	30	23	16	13

Table 10.3: Minimum sample size for the specified ratio t/σ_0 , confidence level p^* , acceptance number c , $\alpha = 2, \gamma = 2$ using Poisson approximation.

p^*	c	t/σ_0						
		1.00	1.25	1.50	1.75	2.00	2.50	3.00
0.75	0	8	5	4	3	2	2	2
0.75	1	15	10	7	5	4	4	3
0.75	2	22	14	10	7	6	5	5
0.75	3	28	18	12	9	8	6	6
0.75	4	35	22	15	12	9	8	7
0.75	5	41	26	18	14	11	9	8
0.75	6	47	30	21	16	13	10	9
0.75	7	53	33	23	18	14	11	10
0.75	8	59	37	26	20	16	13	12
0.75	9	66	41	28	21	17	14	13
0.75	10	72	45	31	23	19	15	14
0.90	0	13	8	6	5	4	3	3
0.90	1	22	14	10	7	6	5	5
0.90	2	30	19	13	10	8	6	6
0.90	3	37	23	16	12	10	8	7
0.90	4	44	28	19	15	12	9	9
0.90	5	51	32	22	17	14	11	10
0.90	6	58	36	25	19	16	12	11
0.90	7	65	41	28	21	17	14	13
0.90	8	71	45	31	23	19	15	14
0.90	9	78	49	34	26	21	16	15
0.90	10	85	53	37	28	22	18	16
0.95	0	17	11	8	6	5	4	4
0.95	1	26	17	12	9	7	6	5
0.95	2	35	22	16	12	9	8	7
0.95	3	43	27	19	14	12	9	8
0.95	4	50	32	22	17	14	11	10
0.95	5	58	36	25	19	15	12	11
0.95	6	65	41	28	21	17	14	13
0.95	7	72	45	31	24	19	15	14
0.95	8	79	50	34	26	21	17	15
0.95	9	86	54	37	28	23	18	17
0.95	10	93	58	40	30	25	20	18
0.99	0	26	16	11	9	7	6	5
0.99	1	37	23	16	12	10	8	7
0.99	2	46	29	20	15	12	10	9
0.99	3	55	35	24	18	15	12	11
0.99	4	64	40	28	21	17	14	12
0.99	5	72	45	31	24	19	15	14
0.99	6	80	50	35	26	21	17	16
0.99	7	89	55	38	29	23	18	17
0.99	8	95	60	42	31	25	20	18
0.99	9	103	64	45	34	27	22	20
0.99	10	110	69	48	36	29	23	21

Table 10.4: Values of the operating characteristic function of the sampling plan $(n, c, t/\sigma_0)$ for a given p^* with $\alpha = 2, \gamma = 2$.

p^*	n	c	t/σ_0	σ/σ_0					
				2	4	6	8	10	12
0.75	21	2	1.00	0.7061	0.9872	0.9981	0.9995	0.9998	0.9999
0.75	13	2	1.25	0.7179	0.9900	0.9987	0.9997	0.9999	0.9999
0.75	9	2	1.50	0.7122	0.9913	0.9990	0.9998	0.9999	0.9999
0.75	6	2	1.75	0.7632	0.9944	0.9994	0.9999	0.9999	0.9999
0.75	5	2	2.00	0.7152	0.9936	0.9993	0.9999	0.9999	0.9999
0.75	4	2	2.50	0.5607	0.9895	0.9991	0.9998	0.9999	0.9999
0.75	3	2	3.00	0.5637	0.9908	0.9993	0.9999	0.9999	0.9999
0.90	28	2	1.00	0.5299	0.9723	0.9957	0.9987	0.9994	0.9997
0.90	17	2	1.25	0.5525	0.9787	0.9972	0.9993	0.9997	0.9999
0.90	11	2	1.50	0.5851	0.9840	0.9981	0.9995	0.9998	0.9999
0.90	8	2	1.75	0.5781	0.9859	0.9985	0.9997	0.9999	0.9999
0.90	6	2	2.00	0.5840	0.9880	0.9988	0.9998	0.9999	0.9999
0.90	4	2	2.50	0.5607	0.9895	0.9991	0.9998	0.9999	0.9999
0.90	3	2	3.00	0.5637	0.9908	0.9993	0.9999	0.9999	0.9999
0.95	33	2	1.00	0.4169	0.9576	0.9931	0.9979	0.9991	0.9995
0.95	20	2	1.25	0.4390	0.9670	0.9954	0.9988	0.9995	0.9998
0.95	13	2	1.50	0.4663	0.9744	0.9969	0.9993	0.9997	0.9999
0.95	9	2	1.75	0.4907	0.9799	0.9977	0.9995	0.9998	0.9999
0.95	7	2	2.00	0.4623	0.9801	0.9980	0.9996	0.9999	0.9999
0.95	5	2	2.50	0.3543	0.9766	0.9977	0.9996	0.9999	0.9999
0.95	4	2	3.00	0.2475	0.9689	0.9973	0.9995	0.9999	0.9999
0.99	43	2	1.00	0.2418	0.9191	0.9857	0.9955	0.9979	0.9988
0.99	26	2	1.25	0.2597	0.9356	0.9903	0.9973	0.9989	0.9995
0.99	17	2	1.50	0.2760	0.9477	0.9930	0.9983	0.9993	0.9997
0.99	12	2	1.75	0.2784	0.9549	0.9945	0.9987	0.9996	0.9999
0.99	9	2	2.00	0.2696	0.9591	0.9954	0.9990	0.9997	0.9999
0.99	6	2	2.50	0.2095	0.9582	0.9978	0.9992	0.9998	0.9999
0.99	4	2	3.00	0.2475	0.9689	0.9973	0.9995	0.9999	0.9999

Table 10.5: Minimum ratio of true σ and required σ_0 for the acceptability of a lot with producer's risk of 0.05 for $\alpha = 2$ and $\gamma = 2$

p*	c	t/ σ_0						
		1.00	1.25	1.50	1.75	2.00	2.50	3.00
0.75	0	0.90	1.13	1.35	1.57	1.80	2.25	2.69
0.75	1	0.89	1.12	1.34	1.55	1.75	2.15	2.56
0.75	2	0.89	1.11	1.33	1.52	1.73	2.13	2.46
0.75	3	0.89	1.11	1.32	1.51	1.72	2.08	2.37
0.75	4	0.89	1.10	1.31	1.50	1.68	2.02	2.33
0.75	5	0.88	1.10	1.30	1.48	1.67	1.98	2.22
0.75	6	0.88	1.09	1.29	1.48	1.66	1.94	2.17
0.75	7	0.88	1.09	1.29	1.47	1.64	1.94	2.13
0.75	8	0.88	1.09	1.28	1.46	1.62	1.92	2.11
0.75	9	0.88	1.09	1.28	1.46	1.62	1.89	2.20
0.75	10	0.88	1.09	1.28	1.45	1.61	1.87	2.18
0.90	0	0.90	1.13	1.35	1.57	1.79	2.25	2.69
0.90	1	0.90	1.12	1.34	1.56	1.77	2.19	2.57
0.90	2	0.89	1.11	1.33	1.55	1.74	2.12	2.46
0.90	3	0.89	1.11	1.32	1.52	1.73	2.08	2.44
0.90	4	0.89	1.10	1.31	1.51	1.70	2.06	2.43
0.90	5	0.89	1.10	1.31	1.50	1.69	2.02	2.37
0.90	6	0.89	1.10	1.31	1.49	1.68	1.99	2.34
0.90	7	0.89	1.10	1.30	1.48	1.67	1.97	2.29
0.90	8	0.88	1.09	1.29	1.48	1.66	1.96	2.24
0.90	9	0.88	1.09	1.29	1.47	1.65	1.95	2.22
0.90	10	0.88	1.09	1.28	1.47	1.63	1.92	2.15
0.95	0	0.90	1.13	1.35	1.58	1.80	2.25	2.68
0.95	1	0.90	1.12	1.34	1.56	1.78	2.19	2.62
0.95	2	0.89	1.11	1.33	1.55	1.75	2.17	2.56
0.95	3	0.89	1.11	1.32	1.53	1.74	2.12	2.50
0.95	4	0.89	1.11	1.32	1.52	1.71	2.06	2.43
0.95	5	0.89	1.10	1.32	1.52	1.70	2.03	2.38
0.95	6	0.89	1.10	1.31	1.51	1.70	2.03	2.33
0.95	7	0.89	1.10	1.31	1.50	1.68	1.99	2.27
0.95	8	0.89	1.10	1.30	1.49	1.67	1.98	2.22
0.95	9	0.88	1.09	1.30	1.48	1.67	1.94	2.20
0.95	10	0.88	1.09	1.29	1.48	1.64	1.97	2.18
0.99	0	0.90	1.13	1.35	1.58	1.80	2.25	2.69
0.99	1	0.90	1.13	1.34	1.56	1.79	2.22	2.63
0.99	2	0.90	1.12	1.34	1.56	1.78	2.18	2.56
0.99	3	0.89	1.11	1.33	1.55	1.75	2.15	2.50
0.99	4	0.89	1.11	1.33	1.53	1.74	2.11	2.47
0.99	5	0.89	1.11	1.32	1.53	1.72	2.09	2.43
0.99	6	0.89	1.11	1.32	1.52	1.71	2.07	2.40
0.99	7	0.89	1.11	1.31	1.51	1.70	2.04	2.36
0.99	8	0.89	1.10	1.31	1.51	1.70	2.01	2.33
0.99	9	0.89	1.10	1.30	1.50	1.68	2.01	2.29
0.99	10	0.89	1.10	1.31	1.49	1.68	1.98	2.27

confidence level p^* , acceptance number c , $\alpha = 2, \gamma = 2$ using Poisson approximation. Table 10.4 gives values of the operating characteristic function of the sampling plan $(n, c, \frac{t}{\sigma_0})$ for given confidence level p^* with $\alpha = 2, \gamma = 2$. Table 10.5 gives Minimum ratio of true σ and required σ_0 for the acceptability of a lot with producer's risk of 0.05 for $\alpha = 2, \gamma = 2$.

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